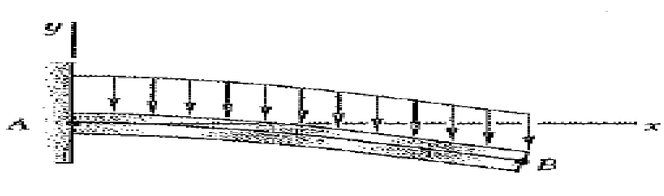
Deflections

Deflection, in structural engineering terms, refers to the movement of a beam or node from its original position due to the forces and loads being applied to the member. Deflection, also known as displacement, can occur from external applied loads or from the weight of the structure itself, and the force of gravity in which this applies. It can occur in beams, trusses, frames and basically any other structure. Also of interest is that knowledge of the deflections is required to analyze indeterminate beams. These are beams in which the number of reactions at the supports exceeds the number of equilibrium equations available to determine these unknowns. The beam subjected to pure bending is bent into an arc of circle and that, within the elastic range, the curvature of the neutral surface can be expressed as:

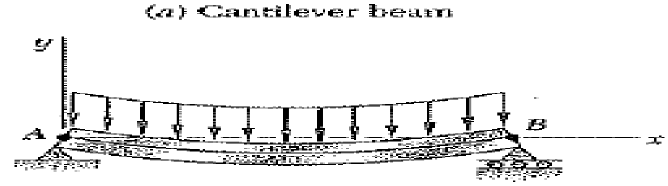
$$\frac{1}{\rho} = \frac{M}{EI}$$

Where *M* is the bending moment, *E is* the modulus of elasticity, and *I* the moment of inertia of the cross section about its neutral axis.
When a beam is subjected to a transverse loading, the previous equation remains valid for any given transverse section, provided that SAIRITVENIMT'S principle applies. However, both the bending moment and the curvature of the neutral surface will vary from section to section. Denoting by *x* the distance of the· section from the left end of the beam, we write:

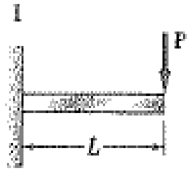
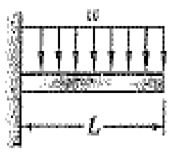
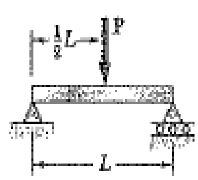
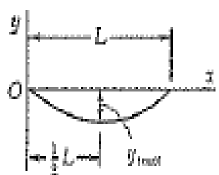
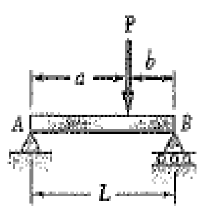
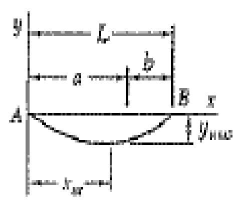
$$\frac{1}{\rho} = \frac{M(x)}{EI}$$



(a) Cantilever beam

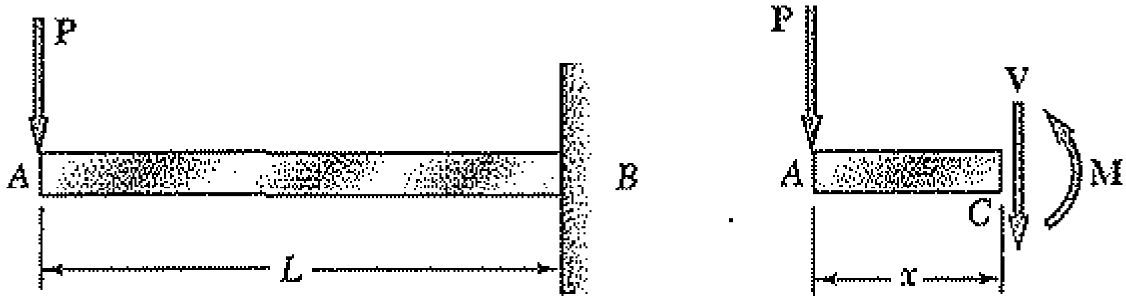


Beam Deflections and Slopes:

Beam and Loading	Elastic Curve	Maximum Deflection	Slope at End	Equation of Elastic Curve
1 		$-\frac{PL^3}{3EI}$	$-\frac{PL^2}{2EI}$	$y = \frac{P}{6EI}(x^3 - 3Lx^2)$
2 		$-\frac{wL^4}{8EI}$	$-\frac{wL^3}{6EI}$	$y = -\frac{w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2)$
3 		$-\frac{ML^2}{2EI}$	$-\frac{ML}{EI}$	$y = -\frac{M}{2EI}x^2$
4 		$-\frac{PL^3}{48EI}$	$\pm\frac{PL^2}{16EI}$	For $x \le \frac{1}{2}L$: $y = \frac{P}{48EI}(4x^3 - 3L^2x)$
5 		For $a > b$: $-\frac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}EIL}$ at $x_m = \sqrt{\frac{L^2 - b^2}{3}}$	$\theta_A = -\frac{Pb(L^2 - b^2)}{6EIL}$ $\theta_B = +\frac{Pa(L^2 - a^2)}{6EIL}$	For $x < a$: $y = \frac{Pb}{6EIL}[x^3 - (L^2 - b^2)x]$ For $x = a$, $y = -\frac{Pa^2b^2}{3EIL}$

Example 1:

The cantilever beam *AB* is of uniform cross section and carries a load **P** at its free end *A*. Determine the equation of the elastic curve and the deflection and. slope at *A*.



Using the free body diagram of the portion AC of the beam, where C is located at a distance *x* from end *A*, we find:

$$M = -Px$$

Then:

$$EI\frac{d^2y}{dx^2} = -Px$$

Integrating in *x*, we obtain

$$EI\frac{dy}{dx} = -\tfrac{1}{2}Px^2 + C_1$$

We now observe that at the fixed end B we have x = *L* and $\theta = dy/dx = 0$
Substituting these values into previous eq. and solving for C1, we have:

$$C_1 = \tfrac{1}{2}PL^2$$

$$EI\frac{dy}{dx} = -\tfrac{1}{2}Px^2 + \tfrac{1}{2}PL^2$$

$$EI\,y = -\tfrac{1}{6}Px^3 + \tfrac{1}{2}PL^2x + C_2$$

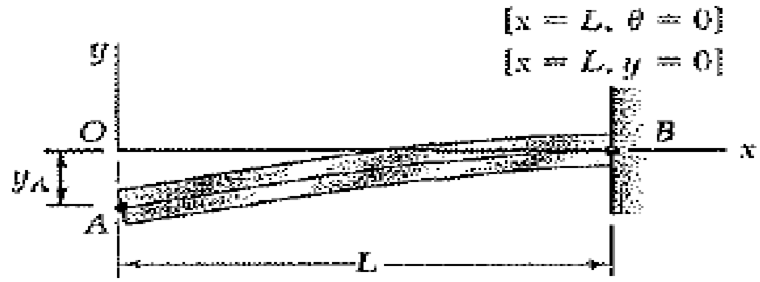
$$0 = -\tfrac{1}{6}PL^3 + \tfrac{1}{2}PL^3 + C_2$$

$$C_2 = -\tfrac{1}{3}PL^3$$

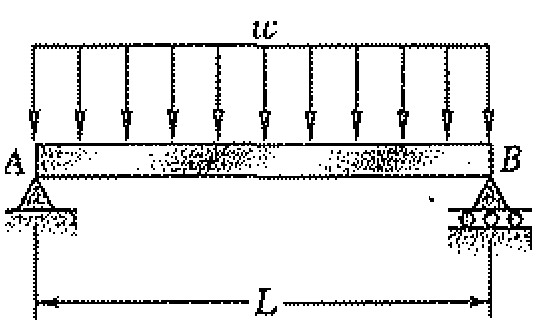
$$EI\,y = -\tfrac{1}{6}Px^3 + \tfrac{1}{2}PL^2x - \tfrac{1}{3}PL^3$$

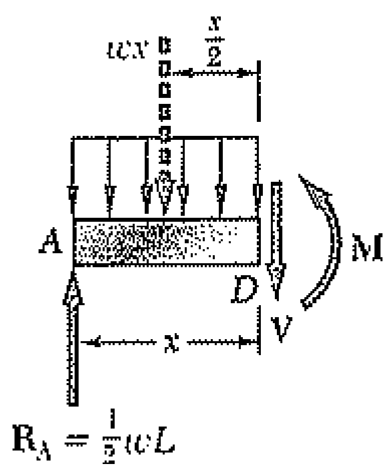
$$y = \frac{P}{6EI}(-x^3 + 3L^2x - 2L^3)$$

$$y_A = -\frac{PL^3}{3EI} \quad \text{and} \quad \theta_A = \left(\frac{dy}{dx}\right)_A = \frac{PL^2}{2EI}$$



Example 2:
The simply supported prismatic beam *AB* carries a uniformly distributed load *w* per unit length.
Determine the equation of the elastic curve and the maximum deflection of the beam.





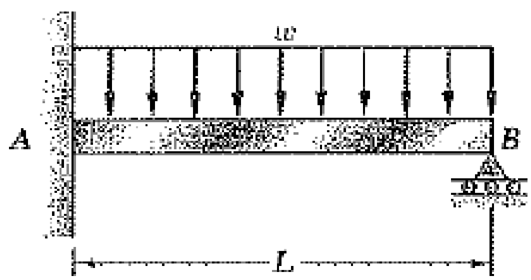
$$y_C = \frac{w}{24EI}\left(-\frac{L^4}{16} + 2L\frac{L^3}{8} - L^3\frac{L}{2}\right) = -\frac{5wL^4}{384EI}$$

$$|y|_{max} = \frac{5wL^4}{384EI}$$

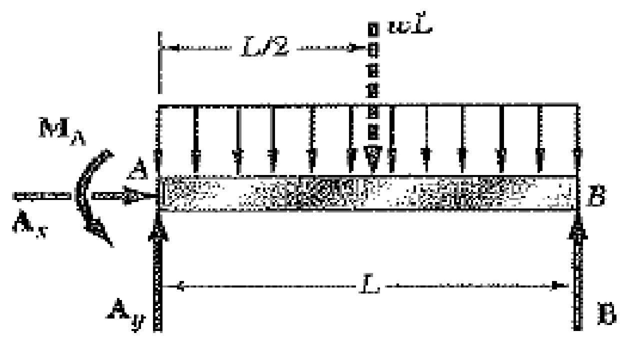
Statically indeterminate beams:

In the preceding sections, our analysis was limited to statically determinate beams. Consider now the prismatic beam AB, which has a fixed end at A and is supported by a roller at B. Drawing the free body diagram of the beam we note that the reactions involve four unknowns, while only three equilibrium equations are available, namely:

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_A = 0$$



(a)

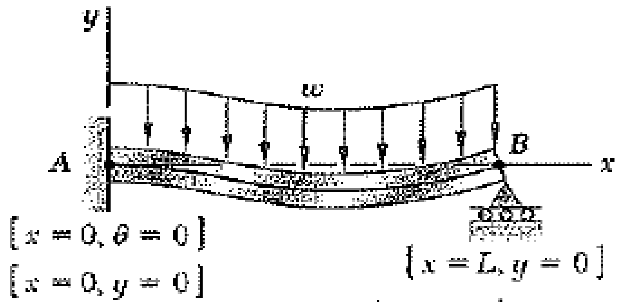


(b)

Since only *Ax* can be determined from these equations, we conclude that the beam is statically indeterminate.

In a statically indeterminate problem, the reactions can be obtained by considering the deformations of the structure involved.

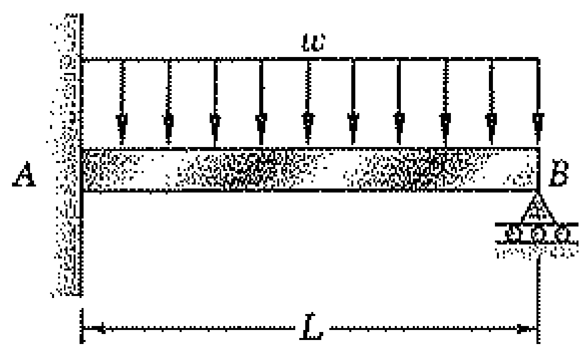
We should, therefore, proceed with the computation of the slope and deformation along the beam. Following the method of equation of the elastic curve, we first express the bending moment *M(x)* at any given point of *AB* in terms of the distance *x* from *A*, the given load, and the unknown reactions Integrating in *x*, we obtain expressions for θ and y which contain two additional unknowns, namely the constants of integration C1 and C2. But altogether six equations are available to determine the reactions and the constants C1 and C2. They are the three equilibrium equations and the three equations expressing that the boundary conditions are satisfied, i.e., that the slope and deflection at A are zero, and that the deflection at *B* is zero.

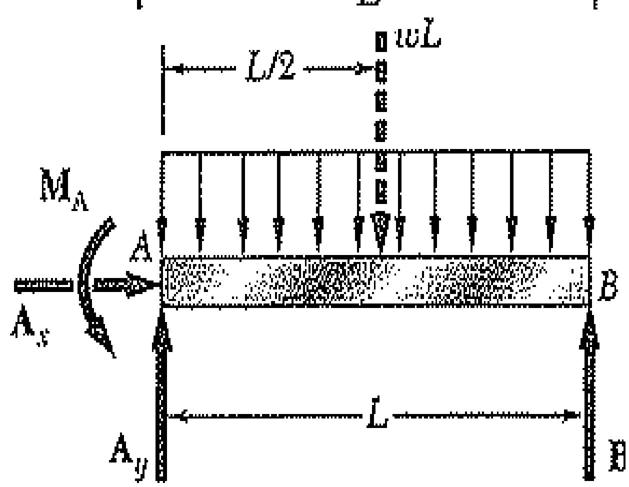


Thus, the reactions at the supports can be determined, and the equation of the elastic curve can be obtained.

Example 3:

Determine the reactions at the supports for the prismatic beam of Fig:





Equilibrium Equations: From the free-body diagram of *we* write

∑ FX = 0 $A_X = 0$

∑ FY = 0 $A_Y + B_Y - wL = 0$

∑ MA=0 $MA + BL - wL^2 = 0$

$$+\circlearrowleft \sum M_C = 0: \quad M + \tfrac{1}{2}wx^2 + M_A - A_y x = 0$$

$$EI\frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + A_y x - M_A$$

$$EI\,\theta = EI\frac{dy}{dx} = -\frac{1}{6}wx^3 + \frac{1}{2}A_y x^2 - M_A x + C_1$$

$$EI\,y = -\frac{1}{24}wx^4 + \frac{1}{6}A_y x^3 - \frac{1}{2}M_A x^2 + C_1 x + C_2$$

We make $x = 0$, $e = 0$, $y = 0$

C1 = C2 = 0. Thus, we rewrite:

$$EI\,y = -\tfrac{1}{24}wx^4 + \tfrac{1}{6}A_y x^3 - \tfrac{1}{2}M_A x^2$$

$$0 = -\tfrac{1}{24}wL^4 + \tfrac{1}{6}A_y L^3 - \tfrac{1}{2}M_A L^2$$

$$3M_A - A_y L + \tfrac{1}{4}wL^2 = 0$$

Solving this equation simultaneously with the three equilibrium equations, we obtain the reactions at the supports:

$$A_x = 0 \qquad A_y = \frac{5}{8}wL \qquad M_A = \frac{1}{8}wL^2 \qquad B = \frac{3}{8}wL$$

Example 2:

The overhanging steel beam *ABC* carries a concentrated load P at end *C*. For portion *AB* of the beam,

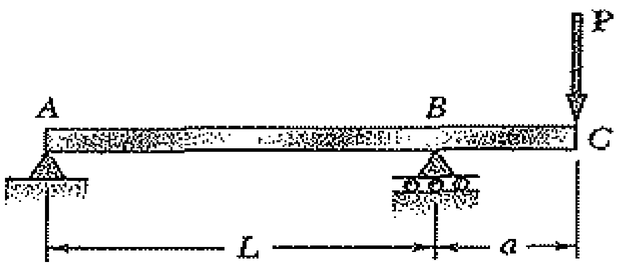
(a) Derive the equation of the elastic curve.

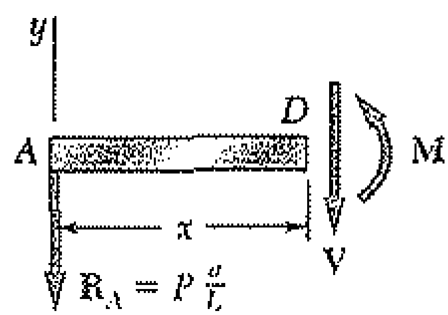
(b) Determine the maximum deflection,

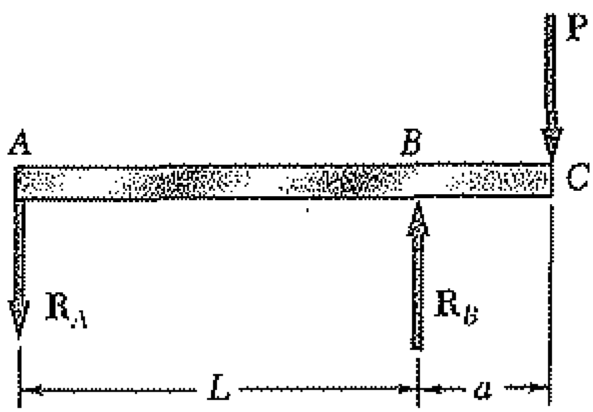
(c) Evaluate Y_{max} for the following data:

$I = 300 * 106\ mm^4$ $\quad E = 200$ GPa $\qquad P = 200$ kN

$L = 4.5$ m $\qquad a = 1.2$ m







$R_A = Pa/L \qquad R_B = P(1 + a/L)$

$$M = -P\frac{a}{L}x \qquad (0 < x < L) \qquad\qquad EI\frac{d^2y}{dx^2} = -P\frac{a}{L}x$$

$$EI\frac{dy}{dx} = -\frac{1}{2}P\frac{a}{L}x^2 + C_1$$

$$EI\,y = -\frac{1}{6}P\frac{a}{L}x^3 + C_1x + C_2$$

X=0 , Y=0 we find C_2=0

X=L , Y=0 we write

$$EI(0) = -\frac{1}{6}P\frac{a}{L}L^3 + C_1L \qquad C_1 = +\frac{1}{6}PaL$$

a) Equation of the Elastic Curve:

$$EI\frac{dy}{dx} = -\frac{1}{2}P\frac{a}{L}x^2 + \frac{1}{6}PaL \qquad \frac{dy}{dx} = \frac{PaL}{6EI}\left[1 - 3\left(\frac{x}{L}\right)^2\right]$$

$$EI\,y = -\frac{1}{6}P\frac{a}{L}x^3 + \frac{1}{6}PaLx \qquad y = \frac{PaL^2}{6EI}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^3\right]$$

b) Maximum Deflection in Portion *AB:*

The maximum deflection Y_{max} occurs at point E where the slope of the elastic curve is zero. Setting *dy/dx* =0

$$0 = \frac{PaL}{6EI}\left[1 - 3\left(\frac{x_m}{L}\right)^2\right] \qquad x_m = \frac{L}{\sqrt{3}} = 0.577L$$

$$y_{max} = \frac{PaL^2}{6EI}[(0.577) - (0.577)^3] \quad y_{max} = 0.0642\,\frac{PaL^2}{EI}$$

c) Evaluation of Y_{max} :

$$y_{max} = 0.0642\,\frac{(200 \times 10^3\,\mathrm{N})(1.2\,\mathrm{m})(4.5\,\mathrm{m})^2}{(200 \times 10^9\,\mathrm{Pa})(300 \times 10^{-6}\,\mathrm{m}^4)} \qquad y_{max} = 5.2\,\mathrm{mm}$$

SHEET 2

For the loading shown, determine:
(a) The equation of the elastic curve for the cantilever beam *AB*
(b) The deflection at the free end
(c) The slope at the free end

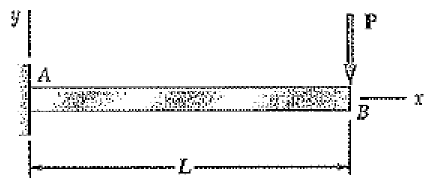


Fig. P9.1

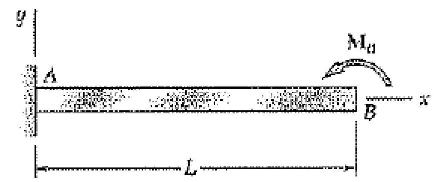


Fig. P9.2

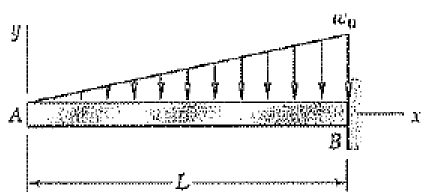


Fig. P9.3

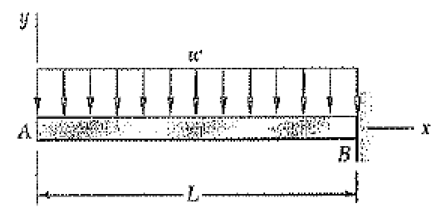


Fig. P9.4

For the cantilever beam and loading shown, determine
(a) The equation of the elastic curve for portion AB of the beam
(b) The deflection at B,
(c) The slope at *B*

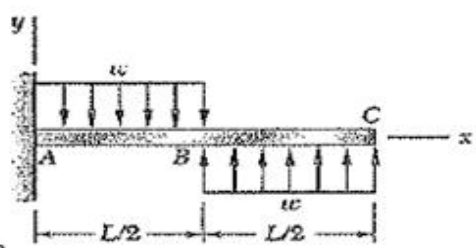


Fig. P9.6

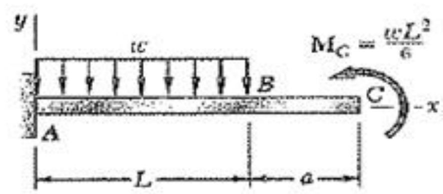


Fig. P9.5