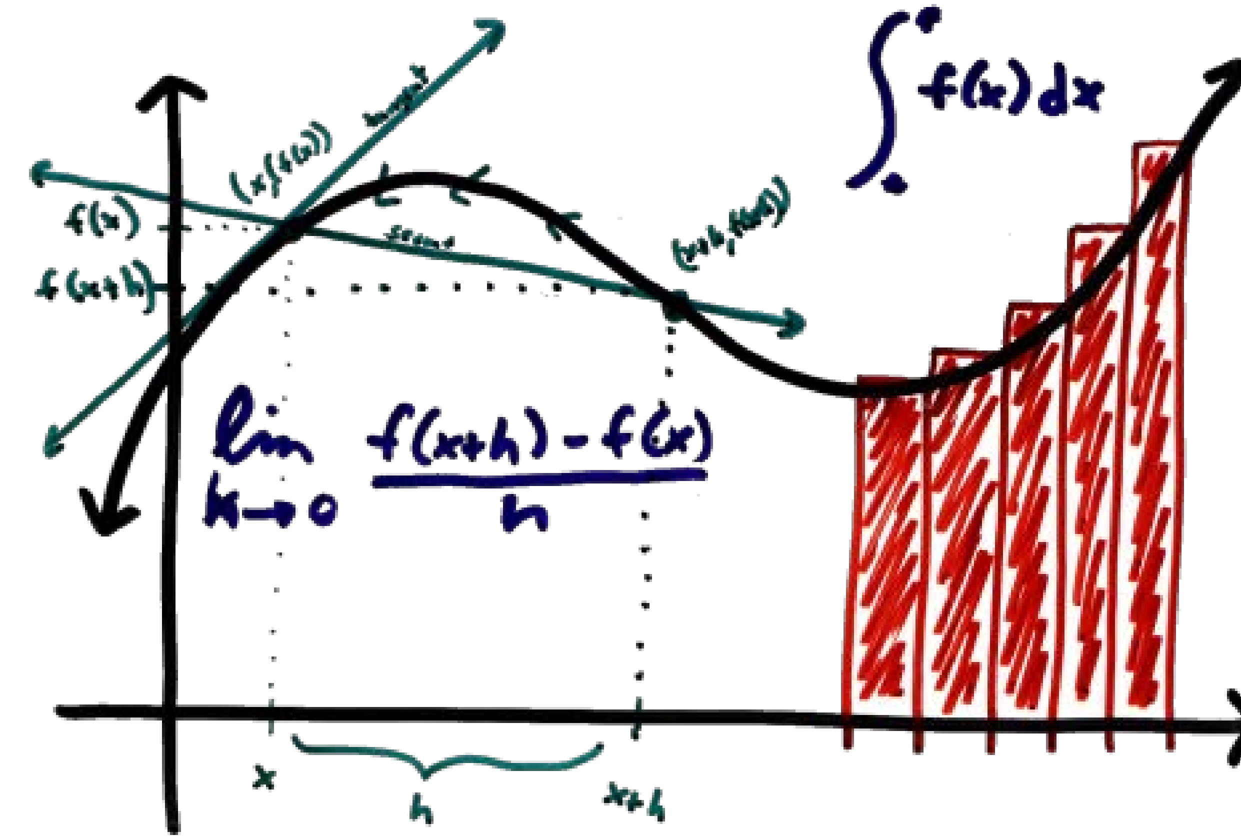
Higher Technological Institute
Tenth of Ramadan City
Computer science department

DIFFERENTIATION & INTEGRATION



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Index

Page

Page

Introduction

This book covers the following topics: real numbers and functions, inequalities and absolute values, the domain and the range of a function, operations and properties of functions, composite functions, even and odd functions, increasing and decreasing functions, the inverse functions, limits and continuity of functions, differentiation (derivatives of exponential, logarithmic, trigonometric, inverse trigonometric functions), integration, definite integrals and applications of the integration, multivariable functions, partial derivatives, extreme values for several variables.

Chapter 1 introduces algebra and functions, chapter two introduces transcendental functions, chapter three introduces limits and continuity, chapter four introduces derivatives and derivatives applications, chapter five introduces inverse trigonometric and inverse hyperbolic functions, and in chapter 6 integrals and integrals applications is introduced.

CHAPTER 1

ALGEBRA AND FUNCTIONS

I- ALGEBRA

This section contains a review of topics from algebra that are prerequisites for calculus. We shall state important facts and work examples without supplying detailed reasons to justify our work.

All concept in calculus are based on properties of the set R of real numbers. There is one-to- one correspondence between R and points on coordinate line (or real number) L as illustrated in Figure 1.1, where 0 is origin. The number 0(zero) is neither positive nor negative

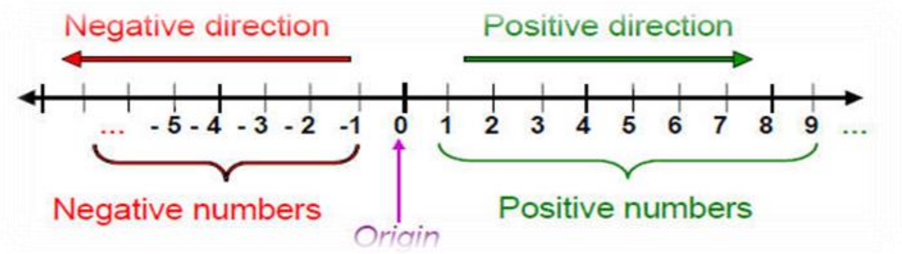

Fig. (1.1)

If a and b are real numbers, then $a > b$ (a is greater than b) if $a - b$ is positive. An equivalent statement is $b < a$ (b is less than a). Referring to the coordinate line in Figure 1.1, we see that $a > b$ if and only if the point corresponding to a lies to the right of the point b. Other types of inequality symbols include $a \le b$ which means $a < b$ or $a = b$, and $a < b \le c$ which means $a < b$ and $b \le c$, for example (*5>3, -7<-2*).

I-1 Properties Of Inequalities

1) If $a > b$ and $b > c$, then $a > c$.
2) If $a > b$ then $a + c > b + c$.
3) If $a > b$ then $a - c > b - c$.
4) If $a > b$ and c is positive, then $ac > bc$
5) If $a > b$ and c is negative , then $ac < bc$

Analogous properties are true if the inequality signs are reversed. Thus, if $a < b$ and $b < c$, then $a < c$, if $a < b$ then $a + c < b + c$, and so on

The absolute value $|a|$ of real number a is defined as follows:

$$|a| = \begin{cases} a & if\ a \geq 0 \\ -a & if\ a < 0 \end{cases}$$

For example $|4| = 4,\ |-5| = 5$

1) $|a| < b$ if and only if $-b < a < b$

I-2 Properties Of The Absolute Value

1) $|a| < b$ if and only if $-b < a < b$

2) $|a| > b$ if and only if $a > b\ or\ a < -b$

3) $|a| = b$ if and only if $a = b\ or\ a = -b$

Definition **:** ***(1.1)***

An equation (in x) is a statement such as:

$x^2 = 3x - 4$, $x^4 + \cos x - \sqrt{x} = 0$

Definition **:** ***(1.2)***

The solutions (or the roots or the values of x satisfy the equations) is a number a that produces a true statement when a is substituted for x. To solve an equation means to find all solutions.

Example **: 1**

Solve each of the following equations (i.e. find the solutions)

a) $x^3 - 2x^2 - 8x = 0$, ***b)*** $2x^2 + 5x^2 - 6 = 0$

Solution:

a) Factoring the left hand side yields

$$x^3 - 2x^2 - 8x = 0 \quad \Rightarrow \quad x(x^2 - 2x \quad - 8) = 0$$

$$\Rightarrow \quad x(x \quad + 2)(x - 4) = 0 \quad \Rightarrow \quad x = 0,\ x \quad + 2 = 0, x - 4 = 0$$

Then the solutions are $0, -2\ and\ 4$.§§§

b) Using the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{with } a = 2, b = 5 \text{ and } c = -6$$

$$\Rightarrow \quad x = \frac{-5 \pm \sqrt{25 - 4(2)(-6)}}{2(2)} = \frac{-5 \pm \sqrt{73}}{4}$$

Hence the solutions are : $\frac{-5}{4} + \frac{\sqrt{73}}{4}$ and $\frac{-5}{4} - \frac{\sqrt{73}}{4}$.§§§

§§§§§§§§§§§§

<u>Definition</u> : (1.3)

An inequality (in x) is a statement that contains at least one of the symbols $<, >, \leq$ *or* $\geq$, such as : $2x - 1 > x^2$ *or* $-5 < 4x + 3 \leq 7$.

The notions of a solution of an inequality and solving an inequality are similar to analogous concepts for equations.

<u>In calculus</u>, we often use intervals. In the definitions that follows, we employ the set notation $\{x: \quad \}$, where the space after colon is used to specify restrictions on the variable x. The notation $\{x: a \leq x \leq b\}$, for example, denote the set of all real numbers greater than a and less than or equal to b- the equivalent interval notation for this set is $(a, b]$. In the following chart, we call (a, b) an open interval, $[a, b]$ a closed interval, $(a, b]$, *and* $[a, b)$ half- open intervals defined in terms of ∞ (infinity) or $-\infty$(negative infinity) infinite intervals.

I-3 <u>Intervals</u>

Intervals

Name of interval	Notation	Inequality description	Number line representation
Finite and closed	[a, b]	a ≤ x ≤ b	a b
Finite and open	(a, b)	a < x < b	a b
Finite and half-open	[a, b)	a ≤ x < b	
	(a, b]	a < x ≤ b	a b
Infinite and closed	(-∞ , b]	-∞ < x ≤ b	b
	[a, +∞)	a ≤ x < +∞	a
Infinite and open	(-∞ , b)	-∞ < x < b	b
	(a, +∞)	a < x < +∞	a
Infinite and open	(-∞ , +∞)	-∞ < x < +∞	

Example : 2

Solve each inequality, and sketch the graph of its solution

a) $-7 \le \frac{3-2x}{2} < 2$, ***b)*** $x^2 - 10 > 3x$

Solution:

a) $-7 \le \frac{3-2x}{2} < 2$

$\Rightarrow \quad -14 \le 3 - 2x < 4$ (multiply by 2)

$\Rightarrow \quad -17 \le -2x < 1$ (subtract 3)

$\Rightarrow \quad -\frac{1}{2} < x \le \frac{17}{2}$ (equivalent inequality)

The solutions are : $(-\frac{1}{2}, \frac{17}{2}]$; the graph is sketched in Figure 1.2 .§§§

Fig. (1.2)

b) $x^2 - 10 \succ 3x$ given

$\Rightarrow \quad x^2 - 3x - 10 > 0$ (Subtract $3x$)

$\Rightarrow \quad (x-5)(x+2) > 0$

The solutions are real numbers in the union $(-\infty, 2) \cup (5, \infty)$ The graph is sketched in Figure 1.3 .§§§

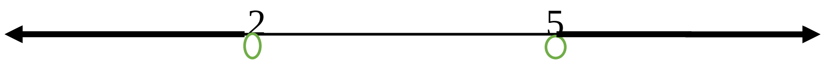

Fig. (1.3)

§§§§§§§§§§§§

Example : 3

Solve each inequality, and then sketch the graph of its solution:

a) $|x - 3| < 5$, ***b)*** $|2x - 7| > 3$

Solution:

a) $|x - 3| < 5 \quad \Rightarrow \quad -5 < x - 3 < 5 \quad \Rightarrow \quad -2 < x < 8$

The solutions are the real numbers in the open interval $(-2,8)$ as shown in Figure 1.4 .§§§

Fig. (1.4)

b) $|2x - 7| > 3$

$\Rightarrow \quad 2x - 7 < -3 \quad or \quad 2x - 7 > 3$

$\Rightarrow \quad 2x < 4 \quad or \quad 2x > 10$

$\Rightarrow \quad x < 2 \quad or \quad x > 5$

The solutions are the real numbers in the open interval $(-\infty, 2) \cup (5, \infty)$ as shown in Figure1.5 .§§§

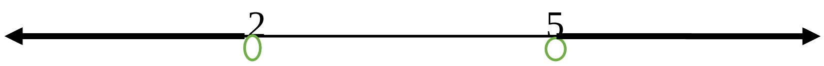

Fig. (1.5)

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A <u>Rectangular Coordinate System</u> is an assignment of ordered pairs (a, b) to points in a plane, as illustrated in Figure1.6. The plane is called a coordinate plane, or xy - plane. Note that in this context (a, b) is an open interval. It should always be clear from our discussion whether (a, b) represent a point or an interval as in Figure1.6.

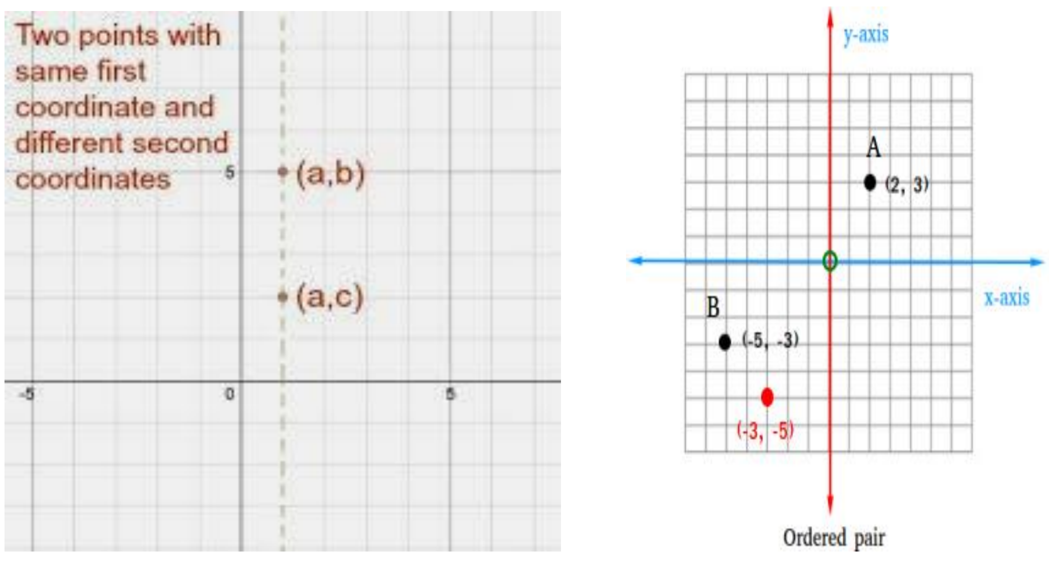

Fig. (1.6)

I-4 Distance Formula

The distance between p_1 and p_2 is :

$$d(p_1,p_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

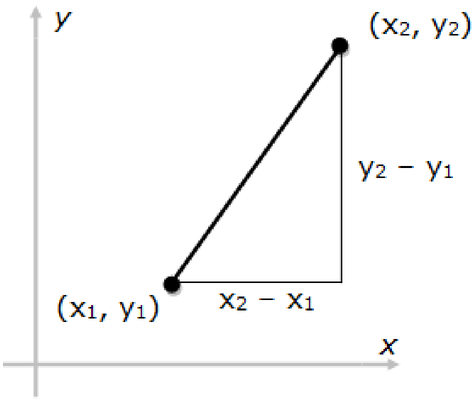

Fig. (1.7)

I-5 The Midpoint Formula

$$M(p_1\ p_2) = M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

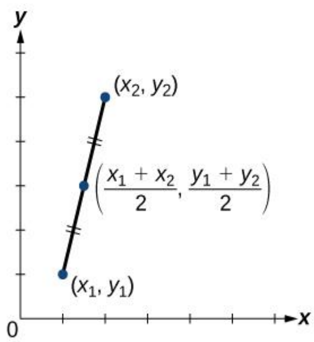

Fig. (1.8)

Example **: 4**

Given $A(4,1) and\ B(10,5)$ find:

a) the distance between $A\ and\ B$, ***b)*** the midpoint of segment AB .

Solution:

a) $d(A,B) = \sqrt{(4+10)^2 + (1+5)^2} = 2\sqrt{58}$

b) $M(AB) = M\left(\frac{4+10}{2}, \frac{1+5}{2}\right) = M(7,3)$.§§§

§§§§§§§§§§§§

<u>An equation</u> in x *and* y is an equality such as:

$$2x + 7y = 12, \quad y = x^2 - 6x + 3 \quad or\ y^2 + sinx = 9$$

A solution is an ordered pair (a, b) that produces a true statement when $x = a$ *and* $y = b$.
The graph of the equation consists of all points (a, b) in a plane that correspond to the solutions.
We shall assume that you have experience in sketching graphs of basic equations in $x,\ y$.

Certain graphs have symmetries as indicated in (Figure1.9), which states tests that can be applied to an equation in $x,\ y$ to determine a symmetry.

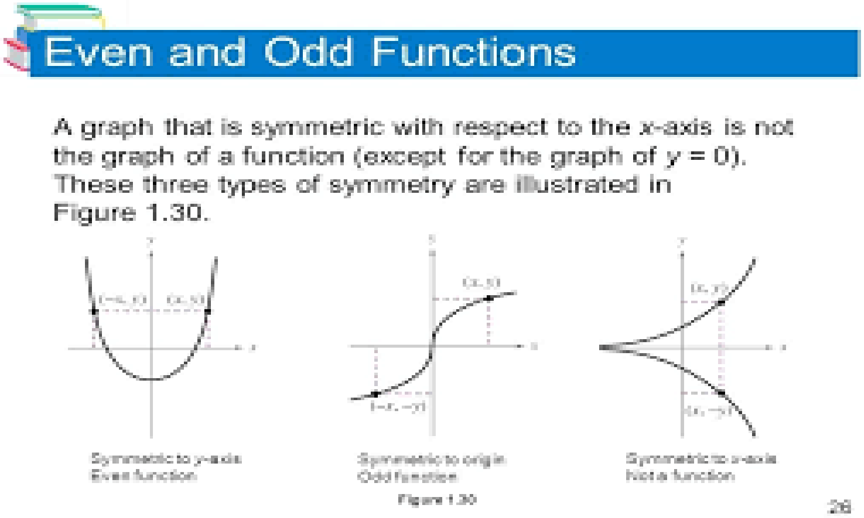

Fig. (1.9)

<u>Example</u> **: 5**

Sketch the graph of

a) $y = \frac{1}{2}x^2$, ***b)*** $y^2 = x$, ***c)*** $y = x^3$.

<u>Solution:</u>

a) By symmetry test the graph $y = \frac{1}{2}x^2$ is symmetric with respect to y −axis. Some points (x, y) on the graph are listed in the given table :

x	0	1	2	3	4
y	0	$\frac{1}{2}$	2	$\frac{9}{2}$	8

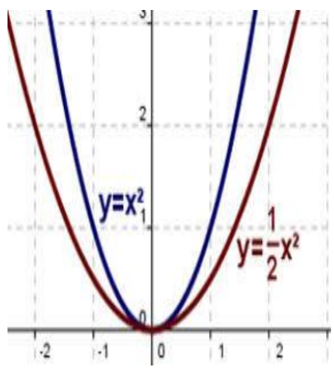

Fig. (1.10)

Plotting, drawing a smooth curve through the points, and then using symmetry gives us the sketch in Figure (1.10) .§§§

b) By symmetry test the graph $y^2 = x$ is symmetric with respect to x −axis. Some points (x, y) on the graph are listed in the following table.

x	0	1	4	9	16
y	0	1	2	3	4

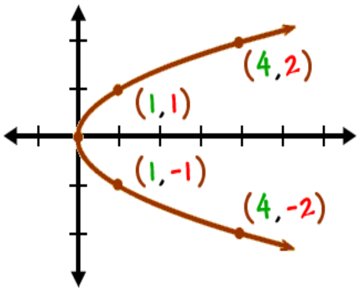

Fig. (1.11)

Plotting, drawing a smooth curve through the points, and then using symmetry gives us the sketch in Figure (1.11) .§§§

c) By symmetry test the graph $y = x^3$ is symmetric with respect to the origin. Some points (x, y) on the graph are listed in the following table.

x	0	1	2	3	4
y	0	1	8	27	64

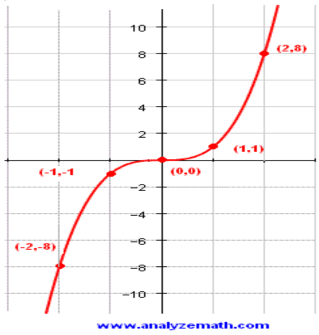

Fig. (1.12)

Plotting, drawing a smooth curve through the points, and then using symmetry gives us the sketch in Figure (1.12)

<u>A circle</u> with center $C(h, k)$ and radius r illustrated in figure 1.12 . If $p(x, y)$ is any point on the circle, then the distance formula $d(p, C) = r$. This leads to the following equation

$$(x - h)^2 + (y - k)^2 = r^2$$

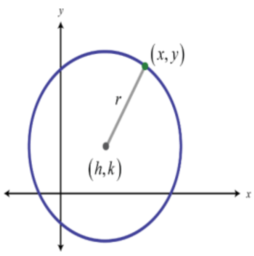

Fig. (1.13)

If $r = 1$, then the circle is called the unit circle. A unit circle U with center at the origin has the equation $x^2 + y^2 = 1$.§§§

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***Example* : 6**

Find the equation of the circle that has the center $C(-1,2)$ straight and passing through the point $p(2,5)$

Solution:

Since the equation of circle is $(x-h)^2 + (y-k)^2 = r^2$ then :

$$(2-3)^2 + (3+1)^2 = r^2 \rightarrow r^2 = 25 \rightarrow r = 5$$.§§§

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I-6 Straight Line

In calculs we often consider lines in a coordinate plane. The following formulas are used for finding there equations :

i) **Slope form** $m = \frac{y_2 - y_1}{x_2 - x_1}$

ii) **Slope intercept form** $y = mx + c$

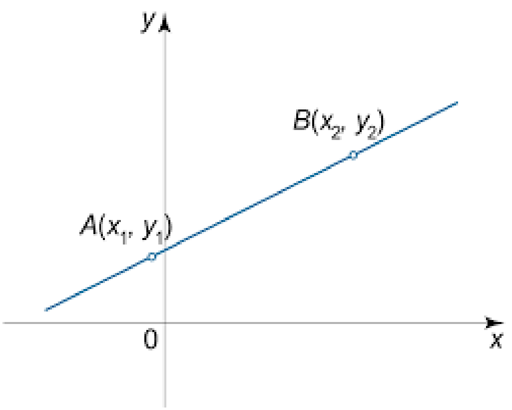

Fig. (1.14)

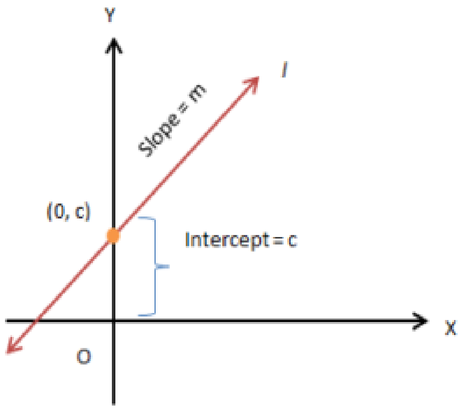

Fig. (1.15)

Some Special Lines

i. Vertical m is undefined and Horizontal $m = 0$

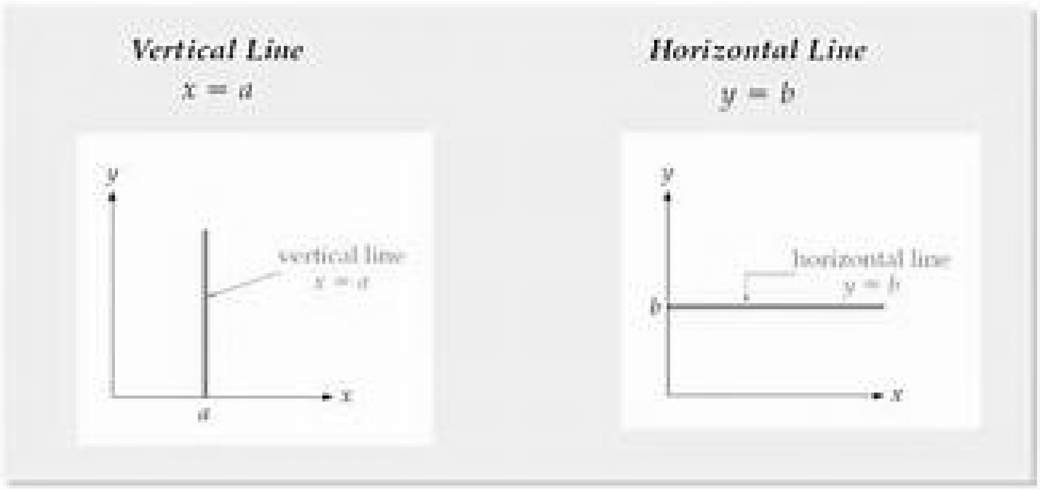

Fig. (1.16)

ii. Parallel $m_1 = m_2$ **iii.** Perpendicular $m_1 . m_2 = -1$

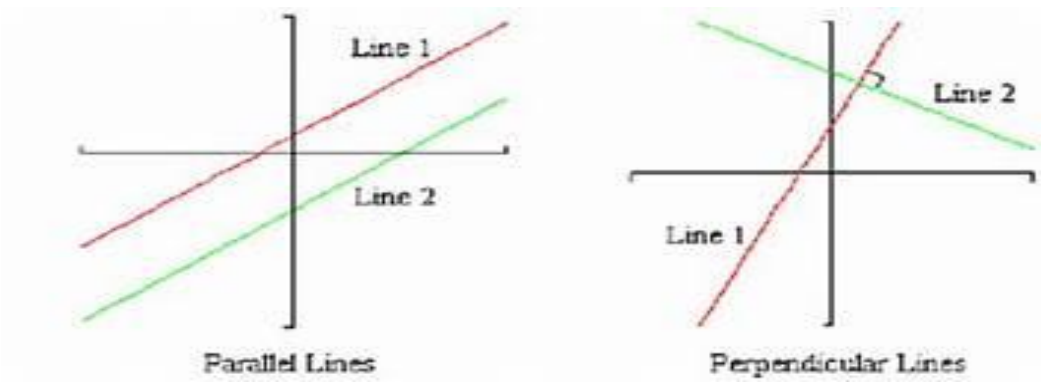

Fig. (1.17)

***Example* : 7**

Find the slope of lines passing through the following points

$a)\ p_1(3,2)\ \ and\ p_2(-1,4)$ $\qquad b)\ p_1(4,3)\ \ and\ p_2(-2,3)$

$c)\ p_1(4,4)\ \ and\ p_2(-1,4)$

Solution:

a) $m = \frac{4-2}{-1-3} = -\frac{1}{2}$.

b) $m = \frac{3-3}{4-(-2)} = 0$ (i.e. the line is horizontal) .

c) $m = \frac{4-(-1)}{4-4} = \frac{5}{0}$ (i.e. the line is vertical) . .§§§

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A linear equation: in $x\ and\ y$ is an equation of form $a\,x + b\,y + d = 0$

a and b not both zero .

***Example* : 8**

Find a linear equation for line through $A = (4,7)and\ \ B = (-3,2)$

Solution:

The slope of the line is $m = \frac{7-2}{-3-4} = -\frac{5}{7}$

We may use A or B for (x_1, y_1) in the point slope $m = \frac{y - y_1}{x - x_1}$

$-\frac{5}{7} = \frac{y-7}{x-4} \Rightarrow -5x + 20 = 7y - 49 \Rightarrow 7y + 5x - 69 = 0$.§§§

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***Example* : 9**

a) Find the slope of the line l with equation : $2x - 5y = 9$.

b) Find the equations for the lines through $p(3,-4)$ that are parallel to l and perpendicular to l .

Solution:

a) Rewrite the equation as: $5y = 2x - 9$ divided both sides by 5, we obtain $y = \frac{2}{5}x - \frac{9}{5}$, and then the slope is $m = \frac{2}{5}$.§§§

b) If the line is parallel to l then $m = m_1 = \frac{2}{5}$,

The equation of line passing through $p(3, -4)$ is

$$y - y_1 = m(x - x_1) \Rightarrow y + 4 = \frac{2}{5}(x - 3)$$

$$5y + 20 = 2x - 6 \Rightarrow 2x - 5y = 26$$.§§§

If the lines are perpendicular to l then

$$m.m_1 = -1 \rightarrow m_1 = -\frac{5}{2}$$

The equation of line passing through $p(3, -4)$ is

$$y - y_1 = m(x - x_1)$$

$$y + 4 = -\frac{5}{2}(x - 3) \Rightarrow 2y + 8 = -5x + 15$$

$$\Rightarrow \quad 5x + 2y = 7$$.§§§

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Exercise (1-1)

1- Rewrite the expression without using the absolute value symbol

1) $(-5)|3-7|$, 2) $|-6|/(-2)$, 3) $|-7|+|4|$

4) $|4-\pi|$, 5) $|3+x|$ $if\ x<-3$.

2- Solve the following equations:

1) $x^2-5x+6=0$, 2) $3x^2-13x+14=0$.

3) $x^2+3x-5=0$, 4) $x^2+3x-5=0$.

5) $4x^2-3x+12=0$, 6) $x^2+3x-5=0$.

7) $x^2-6x-3=0$.

3- Solve the following inequality :

a) $4x+7<2x-9$, b) $3<\frac{2x-5}{3}<7$.

c) $\frac{x+1}{2x-3}>2$, d) $|3x+4|<5$.

4- Find the distance between *A*, *B* and the midpoint of *AB*

a) A(4,-3), B(6,2) , b) A(-2,-5), B(4,6) .

5- Sketch the graph of the equations :

a) $y=2x^2-1$, b) $y=-x^2+2$.

c) $x=-2y^2$, d) $y=-\sqrt{16-x^2}$.

6- Find an equation of the circle that satisfies the given conditions:

a) Center *C* (-4, 6); passing through *p* (1, 2) .

b) Tangent two both axes; center in the second quadrant; radius 5 .

7- Find an equation of the line that satisfies the given conditions :

a) Through *A* (-1, 4); slope $\frac{2}{3}$, b) *x*-intercept 5, y-intercept -4

c) Through *A* (7,-3); perpendicular to the line $5x-2y=9$.

§§§§§§§§§§§§

II- FUNCTIONS AND GRAPHS

***Definition :* (1.4)**

A function f from a set *D* to a set *E* is correspondence that assigns to each element *x* of the set *D* exactly one element *y* of set *E*.

The element *y* of *E* is the **value** of *f* at *x* and is denoted by *f(x)*, read *f* of *x*. The set *D* is the ***Domain*** of the function *f*, and the set *E* is the ***Co-Domain*** of *f*. The range of *f* is the subset of co-domain *E* consisting of all possible function values f(x) for *x* in *D***.**

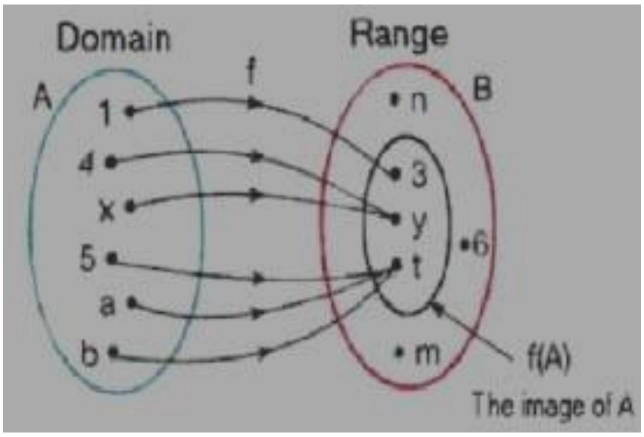


Fig. (1.18)

We sometimes depict functions as shown in Figure (1.18) where the sets *D* and *E* are represented by points within regions in a plane. The curved arrows indicate that the elements f(x), f (a) and f (b) of *E* correspond to the elements *x, a, b* respectively of *D*.

We usually define a function *f by* stating a formula or rule for finding *f(x)*, such as $f(x)=\sqrt{x-2}$. The domain is then assumed to be the set of all real number such that $f(x)$ is real. Thus, for $f(x)=\sqrt{x-2}$ the domain is infinite interval $[2,\infty)$. If x is in the domain, we say $f(x)$ is defined at x, or $f(x)$ is exist. If S is a subset of the domain, then $f(x)$ is defined on S. The terminology $f(x)$ is undefined at x means that x is not in the domain of $f(x)$.

Example **: 10**

Let $f(x) = \frac{\sqrt{x+5}}{2-x}$

a) Find the domain of f , b) Find $f(6), f(-1), f(-a) and -f(a)$

Solution:

a) Note that $f(x)$ is a real number if and only if the radicand $x+5$ is non-negative and denominator $2-x$ is not equal 0. Thus, $f(x)$ exists if and only if (iff)

$$x+5 \geq 0 \text{ and } 2-x \neq 0$$

Or, equivalent, $x \geq -5$ and $x \neq 2$

Hence, the domain is $[-5,\infty) \cup (2,\infty)$

b) $f(6) = \frac{\sqrt{6+5}}{2-6} = -\frac{\sqrt{11}}{4}$, $f(-16) = \frac{\sqrt{-1+5}}{2+1} = \frac{2}{3}$

$f(-a) = \frac{\sqrt{-a+5}}{2+a}$, $-f(a) = \frac{\sqrt{a+5}}{a-2}$

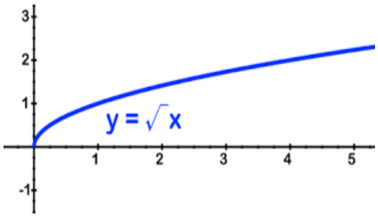
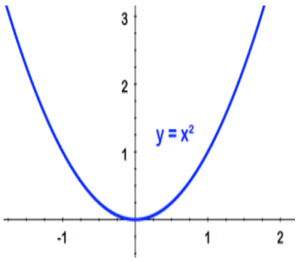
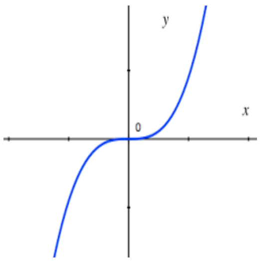
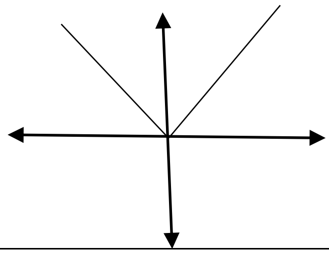
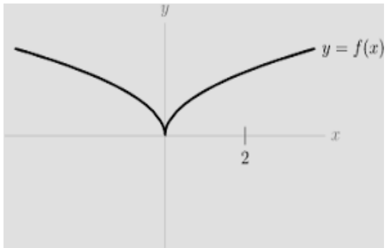
II Functions

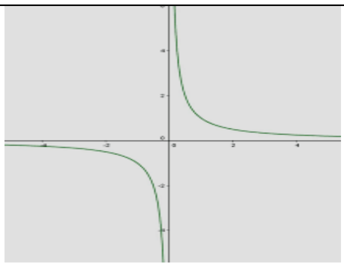
II-1 Even And Odd

** $f(x)$ is an even function if $f(x) = f(-x)$ for every x in the domain of $f(x)$ then the graph of $f(x)$ is symmetric with respect to y-axis .

** $f(x)$ is an odd function if $f(-x) = -f(x)$ for every x in the domain of $f(x)$ then the graph of $f(x)$ is symmetric with respect to the origin.

The next illustration contains Sketches of graphs of some common functions and indicates the symmetry, the domain, and the range for each.

Function	**Graph**	**Symmetry**	**Domain D and rang R**
$f(x) = \sqrt{x}$		None	$D = [0, \infty)$ $R = [0, \infty)$
$f(x) = x^2$		*y*-axis even function	$D = (-\infty, \infty)$ $R = [0, \infty)$
$f(x) = x^3$		Origin Odd function	$D = (-\infty, \infty)$ $R = (-\infty, \infty)$
$f(x) = \|x\|$		*y*-axis even function	$D = (-\infty, \infty)$ $R = [0, \infty)$
$f(x) = x^{\frac{2}{3}}$		*y*-axis even function	$D = (-\infty, \infty)$ $R = [0, \infty)$

$f(x)=\frac{1}{x}$		Origin Odd function	$D=(-\infty,0)\cup(0,\infty)$ $R=(-\infty,0)\cup(0,\infty)$

The piecewise definite function are function described by more than expression, as next example

Example **: 11**

Sketch the graph of the function $f(x)$ defined as follows:

$$f(x)=\begin{cases} x+1 & if\ x<-1 \\ x^2 & if\ -1\le x\le 1, \\ 1 & if\ x>1 \end{cases}$$

Solution:

If $x<-1$, then $f(x)=x+1$ and the graph of f is part of

If $-1\le x\le 1$ then $f(x)=x^2$, and the graph of $f(x)$ is part of $parabola y=x^2$.

If $x>1$, then $f(x)=1$ and the graph is horizontal half line with end point (2, 1)

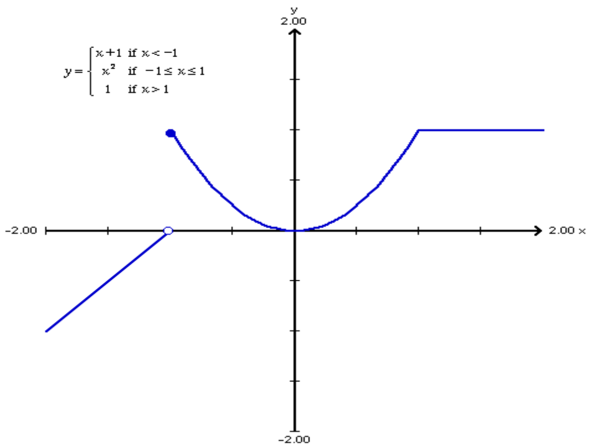


Fig. (1.19)

The greatest integer function $f(x)$ defined by $f(x)=[[x]]$ where $[[x]]$ is the greatest integer less than or equal to x. If we identify R with points on the coordinate line, then $[[x]]$ is the first integer to the left or equal to x.

Example **: 12**

Sketch the graph of the greatest integer function.

Solution:

The x and y coordinates of some points on the graph may be listed as follows :

Value of x	$f(x) = [[x]]$
$-2 \le x < -1$	-2
$-1 \le x < 0$	-1
$0 \le x < 1$	0

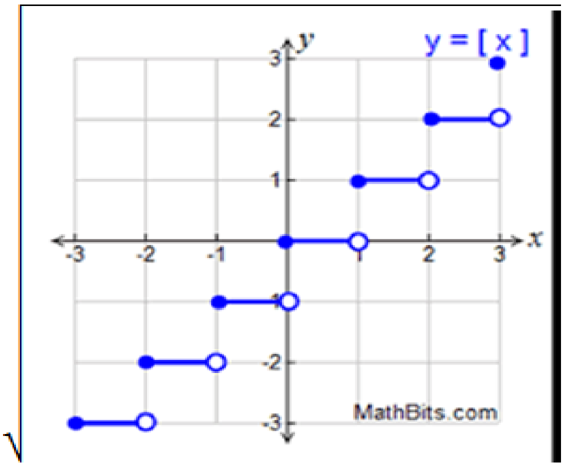


Fig. (1.20)

II-2 Functions Types

1- Polynomial function :

A function $f(x)$ is called a Polynomial function if it has the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers. And ;

* If $a_n \neq 0$, then $f(x)$ has degree n.

* If $f(x) = a$ (called constant function) as $a \neq 0$, then $f(x)$ of degree 0 .

* If $f(x) = ax + b$ (called linear function), then $f(x)$ of degree 1 .

* If $f(x) = ax^2 + bx + c$ (called quadratic function), then $f(x)$ of degree 2 .

2-A rational function:

Is quotient of two polynomial function.

3-An algebraic function:

Is a function that can be expressed in terms of sums, differences, products, quotients, or rational powers of polynomials., for example $f(x) = 5x^5 - 2\sqrt{x} + \frac{x+8}{\sqrt[3]{x^4}}$

4- The Transcendental Functions:

Are functions that are not algebraic. The trigonometric, exponential, and logarithmic functions. In calculus, we often build complicated functions from simpler functions by combining them in various ways, using arithmetic operations and composition.

II-3 Functions Operations

If $f(x)$ and $g(x)$ are functions we defined

** **The sum** $(f+g)(x)=f(x)+g(x)$.

** **The difference** $(f-g)(x)=f(x)-g(x)$.

** **The product** $(fg)(x)=f(x)g(x)$.

****The quotient** $\frac{f}{g}(x)=\frac{f(x)}{g(x)}$.

The domain of $(f+g)(x), f-g$ and fg is the intersection of domains of f and g that is the numbers that are common to both domains. The domain of f/g consists of all numbers x in the intersection such that $g(x)\neq 0$.

Example **: 13**

Let $f(x)=\sqrt{16-x^2}$ and $g(x)=5x+3$. Find the sum, difference, product, and quotient of f and g, and specify the domain of each.

Solution:

The domain of f is $16-x^2\geq 0 \rightarrow x\leq \pm 4$ then $x\in[-4,4]$

The domain of g is R. The intersection of their domains is $[-4,4]$

$(f+g)(x)=\sqrt{16-x^2}+5x+3$.

$(f-g)(x)=\sqrt{16-x^2}-5x-3$.

$(fg)(x)=(\sqrt{16-x^2})(5x+3)$.

$\frac{f}{g}(x)=\frac{\sqrt{16-x^2}}{5x+3}$ and $x\in[-4,4], x\neq -\frac{3}{5}$.

We can also combine two functions to form a new function by the process of composition that is applying one function to the result obtained from the other. Starting with functions f and g, we obtain composite functions $(f\circ g)$ and $(g\circ f)$ read f circle g and g circle f respectively.

***Definition* : (1.5)**

The **composite function** is defined by $(f \circ g)(x) = f(g(x))$, **the domain** of $(f \circ g)$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

***Example* : 14**

If $f(x) = x^2 - 1$ and $g(x) = 3x + 5$, find

a) $(f \circ g)(x)$ and the domain of $(f \circ g)(x)$

b) $(g \circ f)(x)$ and the domain of $(g \circ f)(x)$

Solution:

a) $(f \circ g)(x) = f(g(x))$

$$= f(3x + 5) = (3x + 5)^2 - 1 = 9x^2 + 30x + 24$$

The domain of $(f \circ g)(x)$ is R .§§§

b) $(g \circ f)(x) = g(f(x))$

$$= g(x^2 - 1) = 3(x^2 - 1) + 5 = 3x^2 + 2$$

The domain of $(g \circ f)(x)$ is R .§§§

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***Example* : 15**

If $f(x) = x^2 - 16$ and $g(x) = \sqrt{x}$, find

a) $(f \circ g)(x)$ and the domain of $(f \circ g)(x)$.

b) $(g \circ f)(x)$ and the domain of $(g \circ f)(x)$.

Solution:

a) $(f \circ g)(x) = f(g(x))$

$$= f(\sqrt{x}) = (\sqrt{x})^2 - 16 = x - 16$$

Since the domain of f is R and the domain of g is $x \in [0, \infty)$ then the domain of

$(f \circ g)(x)$ is $x \in [0, \infty)$.§§§

b) $(g \circ f)(x) = g(f(x))$

$$= g(x^2 - 16) = \sqrt{x^2 - 16}$$

To find the domain of $(g \circ f)(x)$ we have $x^2 - 16 \geq 0 \rightarrow x^2 \geq 16 \rightarrow |x| \geq 4$

Then the domain of $(g \circ f)(x)$ is $(-\infty, -4] \cup [4, \infty)$.§§§

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Exercise (1-2)

1) If $f(x)=\sqrt{x-5}-2x$ find $f(5), f(9) and\ f(13)$

2) Find the domain *f*.

a) $f(x)=\frac{x+1}{x^3-4x}$ b) $f(x)=\frac{3x+5}{6x^2+13x-5}$

c) $f(x)=\frac{\sqrt{2x-3}}{x^2-5x+4}$ d) $f(x)=\frac{\sqrt{4x-5}}{x^2-4}$

3) Determine whether *f* is even, odd, or neither even or odd.

a) $f(x)=5x^3+2x$, b) $f(x)=|x|-3$.

c) $f(x)=\sqrt{3x^4+2x^2-5}$, d) $f(x)=x(x-5)$.

4) Sketch the graph of *f*

a) $f(x)=\begin{cases} x+2 & if\ x\le -1 \\ x^3 & if\ |x|<1 \\ -x+3 & if\ x\ge 1 \end{cases}$

b) $(x)=\begin{cases} \frac{x^2-4}{2-x} & if\ x\ne 2 \\ 1 & if\ x=2 \end{cases}$

c) $f(x)=[\![x-3]\!]$, d) $f(x)=[\![x]\!]-3$, e) $f(x)=2[\![x]\!]$

5) Find $(f+g)(x),(f-g)(x),(fg)(x)\ and\ (\frac{f}{g})(x)$ and find its domain.

a) $f(x)=\sqrt{3-2x}$, $g(x)=\sqrt{x+4}$

b) $f(x)=\frac{x}{x-2}$; $g(x)=\frac{3x}{x+4}$

6) Find : $(f\circ g)(x)$ and its domain , $(g\circ f)(x)$ and its domain if :

a) $f(x)=\sqrt{25-x^2}$, $g(x)=\sqrt{x-3}$

b) $f(x)=\frac{x}{3x-2}$; $g(x)=\frac{3}{x}$

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CHAPTER 2

TRANSCENDENTAL FUNCTIONS

I- TRIGNOMATRIC FUNCTIONS

Trigonometry helps us understand angles, triangles, and circles through the use of six special trigonometric functions. In this section, we review some of basic ideas and formulas of trigonometry that are especially important for calculus.

I-1 Angles

An angle is determined by to rays, or line segments, having the same initial point O (the vertex of the angle). If *A* and *B* are points on the rays l_1 *and* l_2 in Figure 2.1, we refer to the angle AOB

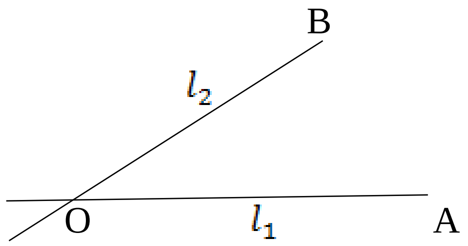

Fig. (2.1)

In a rectangular coordinate system, the standard position of an angle has vertex at the origin and initial side along the positive *x*-axis (see Figure 2.2). A counterclockwise rotation of the initial side produce a Positive angle, where as a clockwise rotation gives negative angle. Lower Greek letters such as α, β and θ are often used to denote angles.

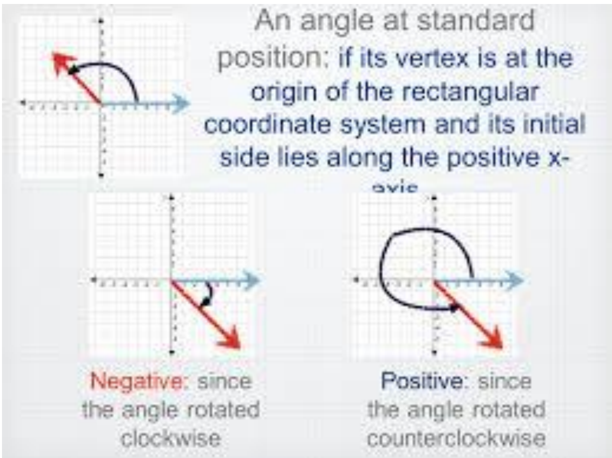

Fig. (2.2)

The magnitude of angle is expressed in either degrees or radians. An angle of degree measure 1^o corresponding to $1/360$ of complete counterclockwise revolution. An angle of radian measure 1 corresponding to $1/2\pi$ of complete counterclockwise revolution. In calculus, the radian is more important unit angular measure. To visualize radian measure, consider a circle of radius 1 with center at vertex of the angle. The radian measure of an angle is the length of the arc the circle that lies between the initial and terminal sides. As in Figure 2.3, θ is an angle of 1 radian

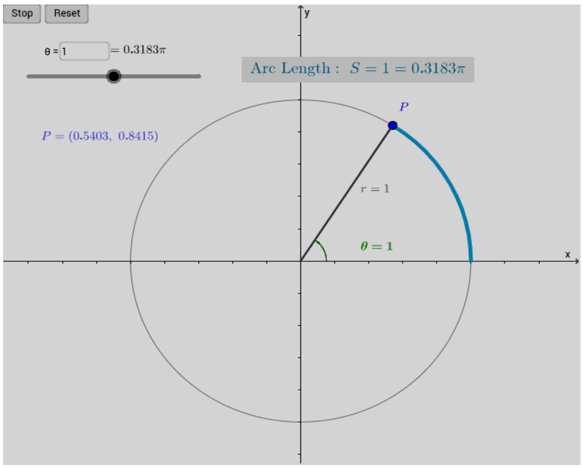

Fig. (2.3)

Since the circumference of the unit circle is 2π, it follows that $2\pi = 360^\circ$

1 radian=$\left(\frac{180}{\pi}\right)^o$ and $1^o \sim 0.01745$ radian.

Conversion between Radians and Degrees

1. To change radian to degrees multiply by $\frac{180}{\pi}$.
2. 1. To change degree to radian multiply by $\frac{\pi}{180}$.

When radian measure of angle is used, no units are indicated. Thus, if an angle θ has radian measures 5, we write$\theta = 5$, but if θ has angle 5^o we write $\theta = 5^o$

Example **: 1**

1) Express $\frac{7\pi}{9}$ radian in degree.
2) Express 105^o degree in radian.

Solution:

1) To convert radian to degree we multiply by $\frac{180}{\pi}$, then

$$\frac{7\pi}{9}\text{ radian}=\left(\frac{7\pi}{9}\right)\left(\frac{180}{\pi}\right)=140^o$$.§§§

2) To convert degree to radian we multiply by $\left(\frac{\pi}{180}\right)$, then

$$105^o=105\left(\frac{\pi}{180}\right)=\frac{7\pi}{12}$$.§§§

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I-2 Trigonometric Functions

The six trigonometric functions are the Sine, Cosine, Tangent, Cosecant, Secant and Cotangent. We denote them by Sin, Cos, Tan, Csc, Sec and Cot respectively. We may define the trigonometric functions in terms of either an angle θ or a real number *x*.

Trigonometric functions of any angle

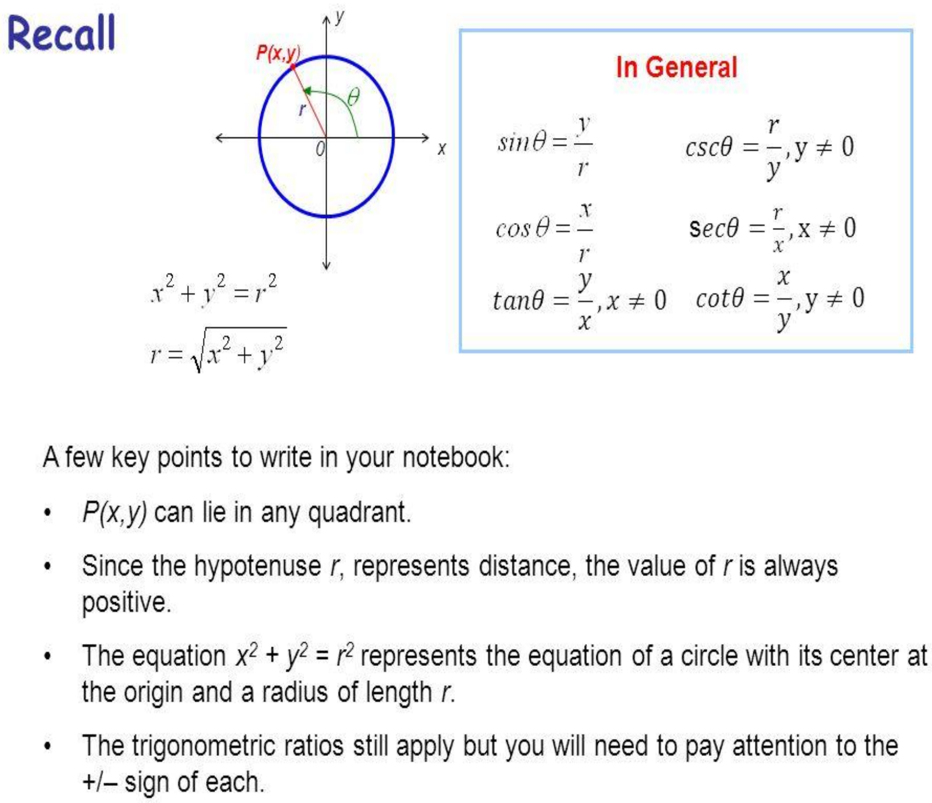
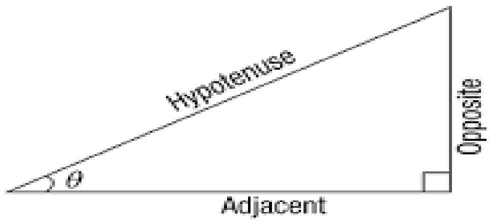


Fig. (2.4)

Trigonometric functions of an acute angle



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

Fig. (2.5)

Trigonometric functions of real number

The value of trigonometric functions at a real number *x* is its value at an angle of *x* radians. From this definition, we see that there is no difference between trigonometric functions of angles measured in radians and trigonometric functions of real numbers. The sign of the value of a trigonometric function of an angle depends on the quadrant containing the terminal side of θ. As shown in Figure 2.6

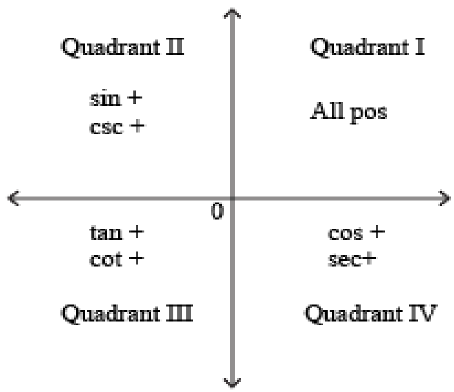


Fig. (2.6)

Example **: 2**

Find the values of trigonometric function for $\theta = \frac{3\pi}{4}$

Solution:

$$\cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}, \qquad \sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}, \qquad \tan\frac{3\pi}{4} = 1$$

$$\sec\frac{3\pi}{4} = -\sqrt{2}, \qquad \csc\frac{3\pi}{4} = \sqrt{2}, \qquad \cot\frac{3\pi}{4} = 1$$

Evaluating Trigonometric functions

Some special value of trigonometric functions

θ Radians	θ Degree	$Sin\theta$	$Cos\theta$	$Sin\theta$	$csc\,\theta$	$sec\,\theta$	$cot\,\theta$
$\frac{\pi}{6}$	30	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	45	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	60	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

Graphs of Trigonometric functions

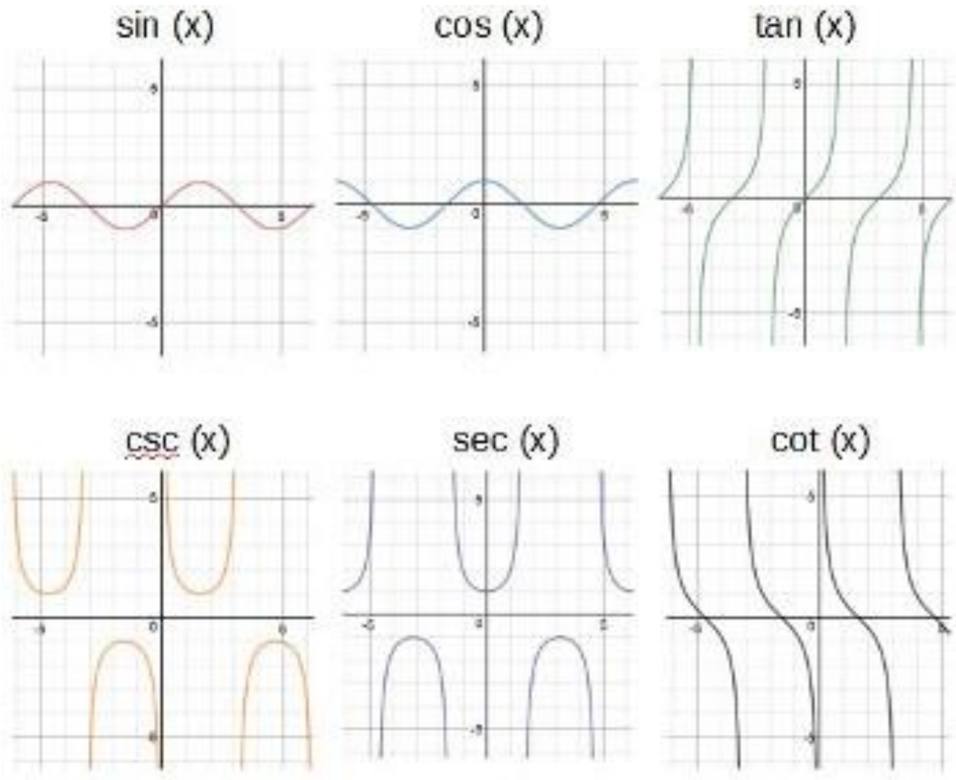


Fig. (2.7)

Trigonometric functions Identities

i) Reciprocal and ratio identities

$$csc\,\theta = \frac{1}{sin\,\theta}\;,\; sec\,\theta = \frac{1}{cos\,\theta}\;,\; cot\,\theta = \frac{1}{tan\,\theta}\;,\; tan\,\theta = \frac{sin\,\theta}{cos\,\theta}\;,\; cot\,\theta = \frac{cos\,\theta}{sin\,\theta}$$

ii) Pythagorean identities

$$cos^2\theta + sin^2\theta = 1 \;,\; 1 + tan^2\theta = sec^2\theta \;,\; 1 + cot^2\theta = csc^2\theta$$

Example **: 3**

Express $\sqrt{16 - x^2}$ in terms of trigonometric function of θ without radicals by making the trigonometric substation $x = 4sinx \;\; for \;\; -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Solution:

$$\sqrt{16 - x^2} = \sqrt{16 - 16sin^2\theta} = 4\sqrt{1 - sin^2\theta} = 4\,cos\theta$$.§§§

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iii) Low of Sines and cosines

If *ABC* is triangle labeled as shown, then the following relationships are true

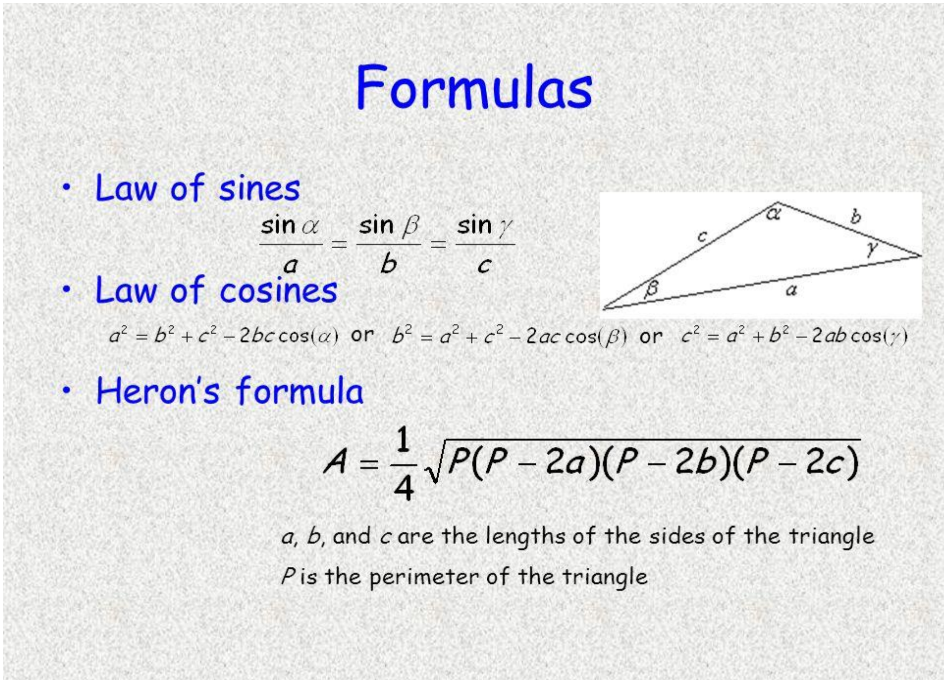

Fig. (2.8)

iv) Additional Trigonometric identities:

** **Formulas for negatives**:

$$\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos\theta, \quad \tan(-\theta) = -\tan\theta$$
$$\csc(-\theta) = -\csc\theta, \quad \sec(-\theta) = \sec\theta, \quad \cot(-\theta) = -\cot\theta$$

** **Addition and subtraction formulas for sine and cosine**:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

For any real numbers α, β

Double angle formulas for sine and cosine

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

Example **: 4**

Verify the following addition formula for tangent function.

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

Solution:

$$\mathrm{Tan}(\alpha + \beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$$

$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

if $\cos\alpha\cos\beta \neq 0$ then we can divide the numerator and the denominator by $\cos\alpha\cos\beta$, we obtain

$$\mathrm{Tan}(\alpha + \beta) = \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{1 - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

Then $$\mathrm{Tan}(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$.§§§

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Trigonometric Equation

Trigonometric Equation is an equation that contains trigonometric expressions. Each fundamental identity is an example of trigonometric equation where every number or angle in the domain of the variable is a solution of the equation.

Example **: 5**

Find the solutions of equation $sin\theta = \frac{1}{2}$ if

a) θ is in the interval $[0,2\pi)$, b) θ is any real number.

Solution:

a) If $sin\theta = \frac{1}{2}$, then $\theta = \frac{\pi}{6}$. If we regard θ as an angle in standard position, then, since $sin\theta > 0$ the terminal side is in either quadrant I or II, thus $\theta = \frac{\pi}{6}$ and $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$. §§§

b) Since the sine function has period 2π then

$\theta = \frac{\pi}{6} + 2n\pi$ and $\theta = \frac{5\pi}{6} + 2n\pi$ for every integer *n*. 2. .§§§

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Exercise (2-1)

I. Find the exact radian measure of the angle:

1) 150^o , 2) 120^o , 3) 450^o , 4) -135^o .

5) 225^o , 6) 210^o , 7) 630^o , 8) -60^o .

II. Find the exact degree measure of the angle:

1) $\frac{2\pi}{3}$, 2) $\frac{5\pi}{6}$, 3) $\frac{3\pi}{4}$, 4) $-\frac{7\pi}{2}$.

5) $\frac{11\pi}{6}$, 6) $\frac{4\pi}{3}$, 7) $\frac{11\pi}{4}$, 8) $-\frac{5\pi}{3}$.

III. Solve the following equation

1) $\sin\theta = \frac{3}{5}$, 2) $\cot\theta = 1$, 3) $\cos\theta = -\frac{1}{\sqrt{2}}$.

IV. Express the following expression in terms of trigonometric functions

1) $\sqrt{16 - x^2}$, $x = 4cos\theta$ for $-\pi \le \theta \le \pi$.

2) $\frac{\sqrt{x^2+4}}{x^2}$, $x = 2tan\theta$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

V. Verify the identity

1) $(1 - sin^2 t)(1 + tan^2 t) = 1$, 2) $\frac{1}{csc\theta - cot\theta} = csc\theta + cot\theta$

3) $2sin^2 2t + \cos 4t = 1$

VI. Find all solutions of equation.

1) $2\cos 2\theta - \sqrt{3} = 0$, 2) $2\sin 3\theta + \sqrt{2} = 0$

VII. Find all solutions of equations in $[0, 2\pi]$.

1) $2sin^2 u = 1 - \sin u$, 2) $\sin x + \cos x \cot t = cscx$

3) $\sin\frac{1}{2}u + \cos u = 1$

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II- EXPONENTIAL AND LOGARITHMS

Exponential and Logarithms functions play a major role in calculus. We defer a complete rigorous definition of exponential and logarithmic functions until we have developed the necessary tools of calculus. We review some of their properties in this section.

II-1 Exponential Functions

The exponential function with base b is defined by $f(x) = b^x$, wand x is any real number.

If x is negative integer, then $x = -n$ for some positive integer n and a^n and $b^x = b^{-n} = \frac{1}{b^n}$

If x is a rational number of the form $= \frac{m}{n}$, where m and n are integers with $n > 0$, then $b^x = b^{\frac{m}{n}} = (\sqrt[n]{b})^m$

In the graphs of $f(x) = b^x$ shown in Figure 2.9

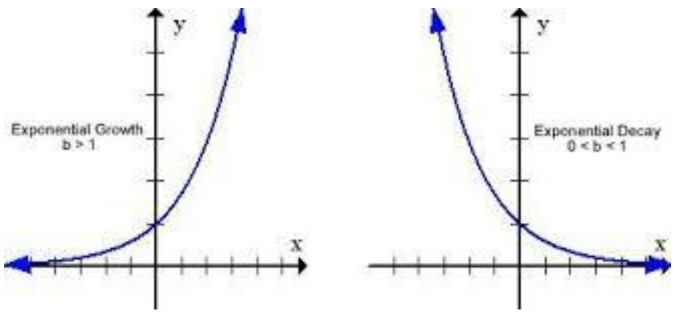

Fig. (2.9)

Properties of exponential functions

The exponential function $f(x) = b^x$ is one to one, that is, for any real numbers x_1 and x_2:

1- If $x_1 \neq x_2$ then $b^{x_1} \neq b^{x_2}$

2- If $b^{x_1} = b^{x_2}$ then $x_1 = x_2$

Laws of exponential function

If u and v are any two real numbers, then

1) $b^u b^v = b^{u+v}$, 2) $\frac{b^u}{b^v} = b^{u-v}$, 3) $(b^u)^v = b^{u\,v}$

An exponential equation is an equation involving exponential functions. We can often solve exponential equation by using the one-to-one property.

Example **: 6**

Solve the exponential equation $5^{5x} = 5^{3x-4}$.

Solution:

By using (1.1) then $5x = 3x - 4, \Rightarrow 2x = -4 \Rightarrow x = -2.$ §§§

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Example **: 7**

Solve the exponential equation $3^{5x-2} = 9^{2x+3}$

Solution:

By using **(1.2)** then $9^{2x-3} = 3^{4x+6} \Rightarrow 3^{5x-2} = 3^{4x+6},$

$\Rightarrow 5x - 2 = 4x + 6 \Rightarrow x = 8$. §§§

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II-2 Logarithmic Functions

If b is a positive number (other than 1), then the exponential function with base b is one-two one function whose range is the set of positive real numbers. Thus, given a positive number *x*, there will be unique number *y* such that $x = b^y$.

The number *y* is called the logarithm of *x* with base *b*. We denote this number as $log_b\, x$.

Definition : *(2.1)*

If b is a positive number (other than 1), then the logarithm of *x* with base *b* is defined by $y = log_b\, x$ if and only if $x = b^y$ for every *x>0*, and every real number *y*.

We call first the **logarithmic form** and second **the exponential form**. Consider the following equivalent forms.

The following table given a numerical example to clarify the definition :

logarithmic form	exponential form
$log_5 u = 2$	$5^2 = u$
$log_b 8 = 3$	$b^3 = 8$
$log_q p = r$	$q^r = p$
$log_4 (2t + 3) = w$	$4^w = 2t + 3$

Properties of logarithms and evaluating exponential forms

Property of $log_b x$ form	exponential form
$log_b 1 = 0$	$b^0 = 1$
$log_b b = 1$	$b^1 = b$
$log_b b^x = x$	$b^x = b^x$
$log_b x = log_b x$	$b^{log_b x} = x$

To obtain graphs of logarithmic functions, we first show that $log_b x$ and b^x are inverse of each other. The graph of either function is the reflection of the graph of the other cross the line *y=x*. shows typical graphs of these functions for *b>1*.

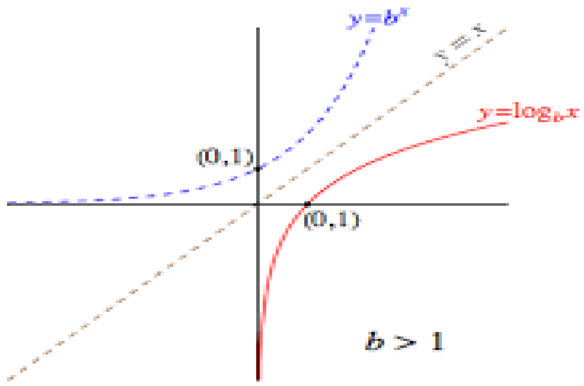

Fig. (2.10)

Properties of Logarithmic functions

The logarithmic function f given by $f(x) = log_a x$ is one-to-one, that is, for any two positive real numbers x_1 and x_2:

If $x_1 \neq x_2$ then $log_a x_1 \neq log_a x_2$

If $log_a x_1 \neq log_a x_2$ then $x_1 = x_2$

Lows of logarithms

If u and v are any two positive real numbers, then

$log_b(u\,v) = log_b u + log_b v$, $log_b\left(\frac{u}{v}\right) = log_b u - log_b v$, $log_b(u)^c = c\,log_b u$

for every real number *c*.

Logarithms with base 10 are called common logarithms. And the symbol $\log x$ in abbreviation for $log_{10} x$. A second widely used logarithm is the natural logarithm, denoted $\ln x$, which has the irrational number *e* for its base.

Change of base formula logarithms

If *x>0* and if *a* and *b* are positive real numbers other than 1, then $log_b\, x = \frac{log_a x}{log_a b}$

Example **: 8**

Approximate $log_7 32$ using common logarithms.

Solution:

Using change of base formula logarithms with *a=10* we have

$$log_7 32 = \frac{log\ \ 32}{log\ \ 7} = 1.7810 \cdot §§§$$

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A logarithmic equation: is an equation involving logarithmic functions. We can often solve logarithmic equations by using the one-to-one property of logarithmic functions.

Example **: 9**

Solve the logarithmic equation $log_5(5x-3) = log_5(3x+7)$

Solution:

By using the one-to-one property, then $5x - 3 = 3x + 7 \rightarrow x = 5$. §§§

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Exercise (2-2)

1-Sketch the graph of f

a) $f(x) = 2^x$, b) $f(x) = 4^x + 5$, c) $f(x) = 2^{-x}$.

2-Solve the following equations

a) $5^{2x+5} = 5^{7x-10}$, b) $4^{x^2} = 2^{3x+2}$

c) $27^{4-x} = 9^{x-3}$, d) $8^{x-1} = 4^{2x-3}$

3-Change to logarithmic form

a) $5^3 = 125$, b) $m^n = p$, c). $(0.7)^t = \frac{2}{3}$

4-Change to Exponential form

a) $log_2 32 = 5$, b) $log_{10} 1000 = 3$

c) $log_7 m = 5x + 3$, d) $log_a 1994 = 7$

5-Solve the following equation

a) $log_2 x = log_2 \ (8 - x)$, b) $log \ x^2 = \log(-3x - 2)$, c) $log \ x^2 = -4$

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CHAPTER 3

LIMITS AND CONTINUITY

I- LIMITS

I-1 Limits Overviews

The concept of a limit is a central idea that distinguishes calculus from algebra and trigonometry. It is a fundamental to finding the tangent to a curve or the velocity of an object.

Generally in differentiation and integration we interested in the value of $f(x)$ for the function f when x approaches a given number a.
Some times the number a doesn't exist in the domain of the function f that is $f(a)$ is undefined, for example:

Consider the function $: f(x) = \dfrac{x^2 - 1}{x - 1} \quad ; \quad x \neq 1$

We noted that the number 1 doesn't exist in the domain of the function science $f(1) = (0/0)$ which is undefined value. The following table is calculated by the calculator to show the value of the given function when x approaches from 1

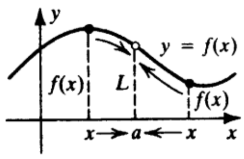
x	$f(x)$	x	$f(x)$
0.9 0.99 0.999 0.9999 0.99999 ↓	1.9 1.99 1.999 1.9999 1.99999 ↓	1.1 1.01 1.001 1.0001 1.00001 ↓	2.1 2.01 2.001 2.0001 2.00001 ↓
1^-	2	1^+	2

We noted that when x approaches from the number 1 the value of the function $f(x)$ approaches from the value 2, which denoted by $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1} = 2$.

Generally if the function f is defined on open interval contain the number a then there exist some questions:
* When x approaches from a, the value of $f(x)$ approaches from a real number L.

** If it is possible for the value of the function $f(x)$ to approach as we want to L by choosing a value for x approaching from a .If the answer for these questions is yes, then we can denoted it by: $\lim_{x \to a} f(x) = L$.

Which means that the point $(x, f(x))$ on the curve of f approaches from the point (a, L) when x approaches from the number a and the following table shows this meaning: the expression

The Expression	The meaning	Graphically
$\lim_{x \to a} f(x) = L$	we can make $f(x)$ approaches from L and chose x approaches from a , $x \neq a$	

<u>There exist three cases for</u> $\underline{f(a)}$ **:**

i) $f(a) = L$, (ii) $f(a) \neq L$, (iii) $f(a)$ doesn't exist .

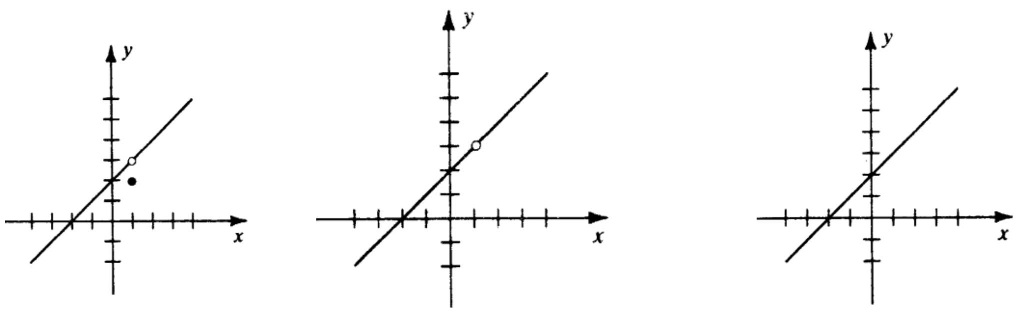
<u>for example</u> |(e.g.)

(i) $f(x) = x + 2$, (ii) $g(x) = \dfrac{x^2 + x - 2}{x - 1}$

(iii) $h(x) = \begin{cases} \dfrac{x^2 + x - 1}{x - 1} & ; \quad x \neq 1 \\ 2 & ; \quad x = 1 \end{cases}$

We noted that $\lim_{x \to 1} f(x) = 3$, $\lim_{x \to 1} g(x) = 3$ and $\lim_{x \to 1} h(x) = 3$

In example (i) $f(1) = 3 = L$, and in example (ii) $g(1)$ doesn't exist whoever the limit is exist, and in example (iii) $h(1) = 2$ which not equal to the limit $L = 3$.



For the function $f(x)=\frac{x^2-1}{x-1}$ it can be simplified as to be $f(x)=\frac{x^2-1}{x-1}=\frac{(x-1)(x+1)}{(x-1)}$:

. $\lim_{x\to1} f(x)=2$ nd a $f(x)=(x+1)$

There exist some functions we may as k is its limit existing or not. as: $\lim_{x\to0}\frac{\sin x}{x}$, if we put $x=0$ we get $(0/0)$ which is undetermined value.
e.g.

► $\lim_{x\to4}(2x-3)=2(4)-3=5$, ► $\lim_{x\to7}\sqrt{x+2}=\sqrt{7+2}=3$

Example **: 1**

If . $\lim_{x\to2} f(x)$ find $f(x)=\frac{2x^2-5x+2}{5x^2-7x-6}$:

Solution **:**

$$\lim_{x\to2} f(x)=\lim_{x\to2}\frac{2x^2-5x+2}{5x^2-7x-6}=\frac{0}{0}$$ which is undetermined value.

By using the factorization to simplify the function and eliminate $(x-2)$ from numerator and denominator

$$\lim_{x\to2} f(x)=\lim_{x\to2}\frac{2x^2-5x+2}{5x^2-7x-6}=\lim_{x\to2}\frac{\cancel{(x-2)}(2x-1)}{\cancel{(x-2)}(5x+3)}$$

$$=\lim_{x\to2}\frac{(2x-1)}{(5x+3)}=\frac{3}{13}$$. §§§

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Example **: 2**

If $f(x)=\frac{x-9}{\sqrt{x}-3}$ then find $\lim_{x\to9} f(x)$ and sketch the function $f(x)$ and find the limit from the sketch.

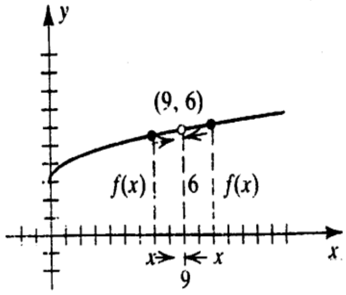
Solution **:**

$$\lim_{x\to9} f(x)=\lim_{x\to9}\frac{x-9}{\sqrt{x}-3}=\frac{0}{0}$$

$$\lim_{x\to9} f(x)=\lim_{x\to9}\frac{x-9}{\sqrt{x}-3}=\lim_{x\to9}\left(\frac{x-9}{\sqrt{x}-3}.\frac{\sqrt{x}+3}{\sqrt{x}+3}\right)\Rightarrow$$

$$=\lim_{x\to9}\left(\frac{(x-9)(\sqrt{x}+3)}{(x-9)}\right)=\lim_{x\to9}(\sqrt{x}+3)=6$$. §§§

To sketch the function $f(x)=\dfrac{x-9}{\sqrt{x}-3}$ Which is the function $y=\sqrt{x}+3$ after simplifying.



The value of $f(x)$ approaches from the value 6 when x approaches from 9. §§§

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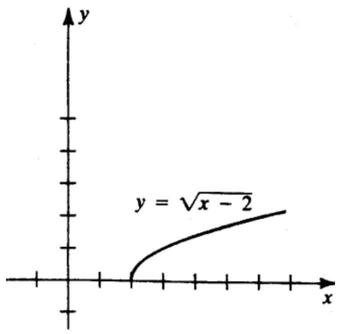
***Example* : 3**

If $f(x)=\sqrt{x-2}$, sketch the graph of f and find if possible:

(i) $\lim\limits_{x\to 2^-} f(x)$ (ii), $\lim\limits_{x\to 2^+} f(x)$, (iii) $\lim\limits_{x\to 2} f(x)$.

***Solution* :**

The curve as in the following figure:



(i) The right limit $\lim\limits_{x\to 2^+} f(x)$:

If $x>2$ then $x-2>0$ so, $f(x)=\sqrt{x-2}$ is a real value then f is defined so, $\lim\limits_{x\to 2^+}\sqrt{x-2}=\sqrt{2-2}=0$.

(ii) The left limit $\lim\limits_{x\to 2^-} f(x)$:

The left limit doesn't exist where $f(x)=\sqrt{x-2}$ is not real value when $x<2$. (iii) The limit $\lim\limits_{x\to 2} f(x)$ doesn't exist because $f(x)=\sqrt{x-2}$ is undefined on the open interval containing the number 2. §§§

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Theorem **: (3-1)**

$\lim_{x\to a} f(x) = L$ if and only if, i.e. $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$

limit of the function exists if and only if the right and left limits are exist and equal

Example **: 4**

If $f(x) = \frac{|x|}{x}$, sketch the graph of *f* and find:

i) $\lim_{x\to 0^-} f(x)$, (ii) $\lim_{x\to 0^+} f(x)$, (iii) $\lim_{x\to 0} f(x)$

Solution **:**

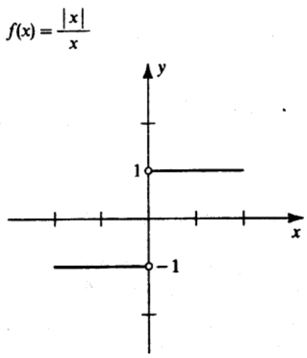
The function is undefined at $x = 0$

(i) If $x < 0$ then $\lim_{x\to 0^-} \frac{|x|}{x} = \frac{-x}{x} = -1$

(ii) If $x > 0$ then $\lim_{x\to 0^-} \frac{|x|}{x} = \frac{x}{x} = 1$

(iii)Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

then, $\lim_{x\to 0} f(x)$ doesn't exist. §§§



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Example **: 5**

If $f(x) = \begin{cases} 3 - x & ; x < 1 \\ 4 & ; x = 1 \\ x^2 + 1 & ; x > 1 \end{cases}$, sketch the graph of *f* and find:

(i) $\lim_{x\to 0^-} f(x)$, (ii) $\lim_{x\to 0^+} f(x)$, (iii) $\lim_{x\to 0} f(x)$

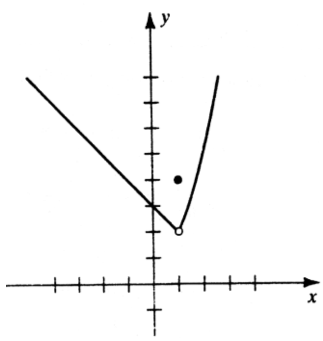
Solution **:**

(i) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (3 - x) = 2$

(ii) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2 + 1) = 2$

Since $\lim_{x\to 1^-} f(x) = 2 = \lim_{x\to 1^+} f(x)$

Then, $\lim_{x\to 1} f(x) = 2$ §§§



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***<u>Example</u>* : 6**

Find $\lim\limits_{x \to 1} f(x)$ if $f(x) = \begin{cases} x+1 & , \ x < 1 \\ 5-2x & , \ x \le 1 \end{cases}$

***<u>Solution</u>* :**

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x+1 = 1+1 = 2 \Rightarrow f(1^-) = 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (5-2x) = 5 - 2.1 = 3 \Rightarrow f(1^+) = 3 .$$

$$f(x) = (x+1) \ \to 1^- \quad 1 \quad 1^+ \leftarrow \ f(x) = (5-2x)$$

Then, $\lim\limits_{x \to 1} f(x)$ doesn't exist . §§§

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***<u>Example</u>* : 7**

Evaluate $\lim\limits_{x \to 3} \dfrac{|x-3|}{x-3}$, if exist .

***<u>Solution</u>* :**

It is noted that : $|x-3| = \begin{cases} (x-3) & , \ x \ge 3 \\ -(x-3) & , \ x < 3 \end{cases}$

$\Rightarrow \quad \lim\limits_{x \to 3^-} f(x) = \dfrac{-(x-3)}{(x-3)} = -1 \quad , \quad \lim\limits_{x \to 3^+} f(x) = \dfrac{(x-3)}{(x-3)} = 1$

$\lim\limits_{x \to 3^+} f(x) \ne \lim\limits_{x \to 3^-} f(x)$ then $\lim\limits_{x \to 3} f(x)$ doesn't exist . §§§

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Exercise (3-1)

(1) Evaluate the following limits if exist:

(a) $\lim_{x\to -1}\frac{x+4}{2x+1}$, (b) $\lim_{x\to 5}\frac{x+2}{x-2}4$, (c). $\lim_{x\to -3}\frac{(x+3)(x-4)}{(x+3)(x+1)}$

(d) $\lim_{x\to 2}\frac{x^2-4}{x-2}$, (e) $\lim_{x\to 4}\frac{x^2-16}{\sqrt{x}-2}$, (f). $\lim_{x\to 1}\frac{x^2-x}{2x^2+5x-7}$

(2) Evaluate the following limits if exist:

$\lim_{x\to a^-} f(x)$, $\lim_{x\to a^+} f(x)$, $\lim_{x\to a} f(x)$.

For the given following functions :

(a) $f(x)=\frac{|x-4|}{x-4}$; $a=4$, (b) $f(x)=\sqrt{5-2x}-x^2$; $a=(5/2)$

(c) $f(x)=\frac{x+5}{|x+5|}$; $a=-5$, (d) $f(x)=\sqrt{5-2x}-x^2$; $a=(5/2)$

(3) sketch the graph for the following functions and evaluate its limits if exist:

(1) $f(x)=\begin{cases} x^2-1 & ;x<1 \\ 4-x & ;x\geq 1\end{cases}$, (2) $f(x)=\begin{cases} x^3 & ;\ x\leq 1 \\ 3-x & ;\ x>1\end{cases}$

(3) $f(x)=\begin{cases} 3x-1 & ;x\leq 1 \\ 3-x & ;x>1\end{cases}$, (4) $f(x)=\begin{cases} |x-1| & ;\ x\neq\leq 1 \\ 1 & ;\ x>=1\end{cases}$

(5) $f(x)=\begin{cases} x^2+1 & ;\ x<1 \\ 1 & ;\ x=1 \\ x+1 & ;\ x>1\end{cases}$, (6) $f(x)=\begin{cases} -x^2 & ;\ x<1 \\ 1 & ;\ x=1 \\ x-21 & ;\ x>1\end{cases}$

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I-2 Limits By Definition

***Definition* : (3-1)**

If the function f is defined on open interval contain the number a , then we call that $f(x)$ ***has a limit*** at the number a and if there exist a real number L such that : $\lim_{x \to a} f(x) = L$, or in other word

For every $\varepsilon > 0$ (called $\varepsilon -$ tolarnce) there exist a $\delta > 0$ (called $\delta -$ tolarnce) and such that :

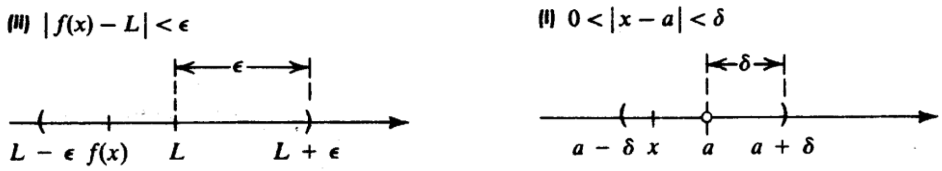
If $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

From definition (3-1):

(i) $|x - a| < \delta \quad \Leftrightarrow . \quad a - \delta < x < a + \delta$

(ii) $L - \varepsilon < y = f(x) < L + \varepsilon \quad \Leftrightarrow \quad |y - L| < \varepsilon$.

and the following figure shows the relation between (i), (ii).



***Example* : 8**

By limit definitions, find the relation between δ , ε , if : $\lim_{x \to 2} \dfrac{2x^2 + x - 10}{(x - 2)} = 9$

***Solution* :**

From the definition of the limits:

$$\left| \frac{2x^2 + x - 10}{(x - 2)} - 9 \right| < \varepsilon \Rightarrow \left| \frac{(2x + 5)(x - 2)}{(x - 2)} - 9 \right| < \varepsilon$$

$$\Rightarrow \quad |(2x + 5) - 9| < \varepsilon \Rightarrow |2x - 2| < \varepsilon \Rightarrow |x - 1| < \frac{\varepsilon}{2}$$

If §§§. $|f(x) - 9| < \varepsilon \quad \forall\ x ; \ 0 < |x - 1| < \delta$; $\delta < \dfrac{\varepsilon}{2}$: then $\delta = (\varepsilon / 2)$

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***Example* : 9**

By limit definitions, prove that: . $\lim_{x \to 3} \frac{x^2 - 5x + 6}{(x-3)} = 1$

***Solution* :**

From the definition of the limits,

$$|f(x) - L| < \varepsilon \quad \Rightarrow \quad \left| \frac{x^2 - 5x + 6}{(x-3)} - 1 \right| < \varepsilon$$

$$\Rightarrow \quad \left| \frac{\cancel{(x-3)}(x-2)}{\cancel{(x-3)}} - 1 \right| < \varepsilon \quad \Rightarrow \quad |(x-2) - 1| < \varepsilon \quad |x-3| < \varepsilon$$

Let $\delta = \varepsilon$ then $, \left| \frac{x^2 - 5x + 6}{(x-3)} - 1 \right| < \varepsilon \ \ \forall x \ ; \ \ 0 < |x - 3| < \delta$

Then the limit is exist and $\lim_{x \to 3} \frac{x^2 - 5x + 6}{(x-3)} = 1$. §§§

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Exercise (3-2)

(1) Use the definition of the limits to show that:

(a) $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$, (b) $\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3} = -5$

(c) $\lim_{x \to 2} (3x - 2) = 4$, (d) $\lim_{x \to -3} (2x + 1) = -5$

(e) $\lim_{x \to -6} (10 - 9x) = 64$, (f) $\lim_{x \to 6} (3 - \frac{1}{2}x) = 0$

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I-3 Limits By Techniques

Theorem : (3-2)

1- If $f(x)$ is defined at a then **:** (a) $\lim\limits_{x \to a} f(x) = f(a)$, (b) $\lim\limits_{x \to a} C = C$

Where *C* is constant. , and if

2- $f(x)$ is a polynomial on the form: $f(x) = c_\circ + c_1 x + c_2 x^2 + ... + c_n x^n$

Then : $\lim\limits_{x \to a} f(x) = c_\circ + c_1 a + c_2 a^2 + ... + c_n a^n$

(e.g.) :

► $\lim\limits_{x \to -\sqrt{2}} 8 = 8$, ► $\lim\limits_{x \to 7} x = 7$

► $\lim\limits_{x \to -3} (5 + 2x - 4x^3) = 5 + 2(-3) + -4(-3)^3 = 5 - 6 - 27 = -28$.

Theorem : (3-3)

(1) $\lim\limits_{x \to a} [f(x)]^m = \left[\lim\limits_{x \to a} f(x)\right]^m$, Where $m \neq 0$ and real.

(2) $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)}$.

(e.g.) :

► $\lim\limits_{x \to 2} (3x^3 - 10x)^3 = \left(\lim\limits_{x \to 2} (3x^3 - 12x)\right)^3 = (3 \times 8 - 10 \times 2)^3 = 4^3$

► $\lim\limits_{x \to 4} \sqrt[3]{x^2 - 6x} = \sqrt[3]{\lim\limits_{x \to 4} (x^2 - 6x)} = \sqrt[3]{4^2 - 6 \times 4} = \sqrt[3]{-8} = -2$.

Theorem : (3-4)

If then , $\lim\limits_{x \to a} g(x) = L_2$, $\lim\limits_{x \to a} f(x) = L_1$; nctions andare two real fu $f(x)$, $g(x)$

(1) $\lim\limits_{x \to a} [f(x) \pm g(x)] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x) = L_1 \pm L_2$

(2) $\lim\limits_{x \to a} [Cf(x)] = C. \lim\limits_{x \to a} f(x) = C.L_1$; $C \in R$

(3) $\lim\limits_{x \to a} [f(x).g(x)] = \lim\limits_{x \to a} f(x). \lim\limits_{x \to a} g(x) = L_1.L_2$

(4) $\lim\limits_{x \to a} \left[\dfrac{f(x)}{g(x)}\right] = \left[\dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}\right] = \dfrac{L_1}{L_2}$; $L_2 \neq 0$

Solved Problems :

1) Evaluate : $\lim_{x\to 2}\frac{2x-3}{x+1}$

Solution :

By direct substitution with $x=2$, we get:§§§. $\lim_{x\to 2}\frac{2x-3}{x+1} \Rightarrow \frac{2.3-3}{2+1}=\frac{1}{3}$

2) Evaluate : $\lim_{x\to 1}\left(\frac{x^2-2x+9}{x^2+2x-2}\right)^{-2/3}$

Solution :

By direct substitution with $x=1$, we get :

$$\lim_{x\to 1}\left(\frac{x^2-2x+9}{x^2+2x-2}\right)^{-2/3} \Rightarrow \left(\frac{\lim_{x\to 1}(x^2-2x+9)}{\lim_{x\to 1}(x^2+2x-2)}\right)^{-2/3}$$

§§§. $\left(\frac{(1^2-2.1+9)}{(1^2+2.1-2)}\right)^{-2/3} = (8)^{-2/3} = \left(2^3\right)^{-2/3} = 2^{-2} = \frac{1}{4} \qquad \Rightarrow$

3) Evaluate $\lim_{x\to(3/2)}\frac{4x^2-9}{2x-3}$:

Solution :

By direct substitution with $x=(3/2)$ we get : $\lim_{x\to(3/2)}\frac{4x^2-9}{2x-3} = \frac{0}{0}$

we must eliminate $x-(3/2)=0 \Rightarrow (2x-3)=0$ from the numerator and denominator

$$\lim_{x\to(3/2)}\frac{4x^2-9}{2x-3} = \lim_{2x\to 3}\frac{\cancel{(2x-3)}(2x+3)}{\cancel{(2x-3)}} = \lim_{2x\to 3}(2x+3) = 6$$.§§§

4) Evaluate $\lim_{x\to 0}\frac{x}{(x+2)^2-4}$:

Solution :

By direct substitution with $x=0$, we get: $\lim_{x\to 0}\frac{x}{(x+2)^2-4}=\frac{0}{0}$

$$\lim_{x\to 0}\frac{x}{(x+2)^2-4} = \lim_{x\to 0}\frac{x}{[(x+2)-2]\,[(x+2)+2]}$$

$$= \lim_{x\to 0}\frac{x}{x\,[(x+\cancel{2})+2]} = \lim_{x\to 0}\frac{1}{[(x+2)+2]} = \frac{1}{4}$$.§§§

5) Evaluate: $\lim_{x \to 0} \dfrac{\sqrt{x+9}-3}{x^2+2x}$

Solution :

By direct substitution with $x=0$, we get : $\lim_{x \to 0} \dfrac{\sqrt{x+9}-3}{x^2+2x} = \dfrac{0}{0}$

$$\lim_{x \to 0} \frac{\sqrt{x+9}-3}{x^2+2x} = \lim_{x \to 0} \frac{\sqrt{x+9}-3}{x^2+2x} \cdot \frac{\sqrt{x+9}+3}{\sqrt{x+9}+3} \quad \Rightarrow$$

$$\lim_{x \to 0} \frac{(\sqrt{x+9})^2-(3)^2}{x(x+2)(\sqrt{x+9}+3)} = \lim_{x \to 0} \frac{\cancel{x}}{\cancel{x}(x+2)(\sqrt{x+9}+3)}$$

§§§ . $\Rightarrow \quad \lim_{x \to 0} \dfrac{1}{(x+2)(\sqrt{x+9}+3)} = \dfrac{1}{2.(\sqrt{9}+3)} = \dfrac{1}{12}$

6) Evaluate: $\lim_{x \to -2} \dfrac{x^4+8x}{x+2}$

Solution :

By direct substitution with $x=-2$, we get: $\lim_{x \to -2} \dfrac{x^4+8x}{x+2} = \dfrac{0}{0}$

$$\lim_{x \to -2} \frac{x^4+8x}{x+2} = \lim_{x \to -2} \frac{x(x^3+8)}{(x+2)}$$

$$= \lim_{x \to -2} \frac{x\cancel{(x+2)}(x^2-2x+4)}{\cancel{(x+2)}} = \lim_{x \to -2} x(x^2-2x+4) = -24$$. §§§

7) Evaluate: $\lim_{x \to 2^+} \dfrac{x^2-4}{\sqrt{x-2}}$

Solution :

By direct substitution with $x = 2^+$, we get: $\lim_{x \to 2^+} \dfrac{x^2-4}{\sqrt{x-2}} = \dfrac{0}{0}$

$$\lim_{x \to 2^+} \frac{x^2-4}{\sqrt{x-2}} = \lim_{x \to 2^+} \frac{x^2-4}{\sqrt{x-2}} \cdot \frac{\sqrt{x-2}}{\sqrt{x-2}}$$

$$\Rightarrow \quad \lim_{x \to 2^+} \frac{(x^2-4)\sqrt{x-2}}{(\sqrt{x-2})^2} = \lim_{x \to 2^+} \frac{[\cancel{(x-2)}(x+2)]\sqrt{x-2}}{\cancel{(x-2)}}$$

§§§ . $\lim_{x \to 2^+} (x+2)\sqrt{x-2} = 2 \times 0 = 0 \quad \Rightarrow$

§§§§§§§§§§

Theorem **: (3-5)**

1) If $f(x)=\dfrac{x^n-a^n}{x-a}$ and $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=\dfrac{0}{0}$ then, the limit of the function (if exist) could be calculated from :

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\quad ;\quad n\in Z\ ,\ n\neq 0\quad ;\quad \text{and}$$

2) If $f(x)=\dfrac{x^n-a^n}{x^m-a^m}$ and $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x^m-a^m}=\dfrac{0}{0}$ then , the limit of the function (if exist) could be calculated from:

$$\lim_{x\to a}\frac{x^n-a^n}{x^m-a^m}=\frac{n}{m}a^{n-m}\quad ;\quad n,m\in Z\ ,\ n,m\neq 0$$

8) Evaluate $\lim\limits_{x\to -2}\dfrac{x^5+32}{x+2}$

Solution **:**

By direct substitution with $x=-2$ we get: $\lim\limits_{x\to -2}\dfrac{x^5+32}{x+2}=\dfrac{0}{0}$

the function can be written as $\lim\limits_{x\to -2}\dfrac{x^5-(-2)^5}{x-(-2)}$

from theorem (3-5) , $n=5$ we get: $\lim\limits_{x\to -2}\dfrac{x^5-(-2)^5}{x-(-2)}=5.(-2)^{5-1}=80$. §§§

9) Evaluate $\lim\limits_{x\to\sqrt{5}}\dfrac{x^7-125\sqrt{5}}{x-\sqrt{5}}$:

Solution **:**

By direct substitution with $x=\sqrt{5}$ we get: $\lim\limits_{x\to\sqrt{5}}\dfrac{x^7-125\sqrt{5}}{x-\sqrt{5}}=\dfrac{0}{0}$

the function can be written as $\lim\limits_{x\to\sqrt{5}}\dfrac{x^7-(\sqrt{5})^7}{x-\sqrt{5}}$,

from theorem (3-5) , $n=7$ we get:

$\lim\limits_{x\to\sqrt{5}}\dfrac{x^7-(\sqrt{5})^7}{x-\sqrt{5}}=7.(\sqrt{5})^{7-1}=7\times 125=875$. §§§

10) Evaluate $\lim_{x\to 1}\dfrac{(x+1)^6-64}{(x+1)^3-8}$:

Solution :

By direct substitution with $x=1$ we get: $\lim_{x\to 1}\dfrac{(x+1)^6-64}{(x+1)^3-8}=\dfrac{0}{0}$

put $y=(x+1)$ then $x\to 1 \Leftrightarrow y\to 2$, then the function could be written as:

$$\lim_{y\to 2}\frac{y^6-(2)^6}{y^3-(2)^3} \quad n=6, m=3$$

$$\Rightarrow \quad \lim_{y\to 2}\frac{y^6-(2)^6}{y^3-(2)^3}=\frac{6}{3}.(2)^{6-3}=16$$.§§§

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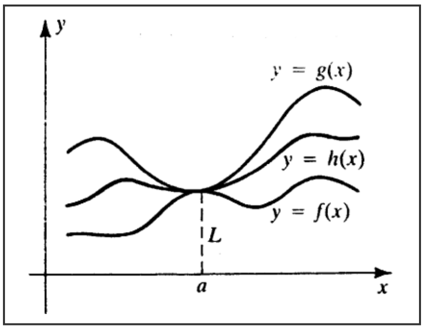
***Theorem* : (3-6) (*Sandwich Theorem*)**

If: $f(x)<h(x)<g(x)$ $\forall x$ in open interval contain a except at a $(x\neq a)$
and if $\lim_{x\to a} f(x)=L=\lim_{x\to a} g(x)$, then $\lim_{x\to a} h(x)=L$

The following figure shows the meaning of the sandwich theorem.



11) Use **sandwich theorem** to prove that: $\lim_{x\to 0} x^2\sin\dfrac{1}{x}=0$

Solution :

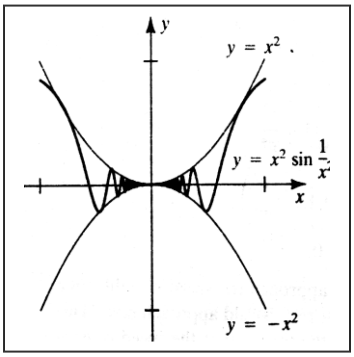
Since $-1\le \sin t\le 1$, $x\neq 0$, $\forall x$, $-1\le \sin\dfrac{1}{x^2}\le 1$ then, $\forall t$ $t\in R$

multiply by (x^2) we get : $-x^2\le x^2.\sin\dfrac{1}{x^2}\le x^2$

which means that the curve of : $y=x^2\sin\dfrac{1}{x^2}$

lies between the two curves $y=x^2$ ، $y=-x^2$:

Since $\underbrace{\lim_{x\to 0}(x^2)=0}_{g(x)}$ ، $\underbrace{\lim_{x\to 0}(-x^2)=0}_{f(x)}$



Then, $\lim_{x\to 0} x^2\sin\dfrac{1}{x}=0$. .§§§

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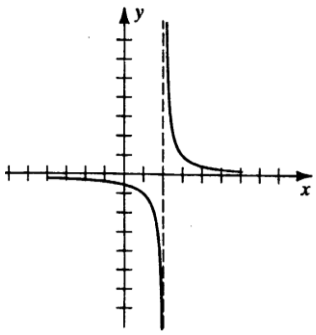
I-4 Limits Involving Infinity

For the two functions $\lim_{x\to a^+} f(x)$ ، $\lim_{x\to a^-} f(x)$ when x approaches from a then, the limit could be increased without maximum limit (or decreasing without minimum limit), to show this

Let $f(x)=\frac{1}{x-2}$ and x approaches from 2, $x>2$ the following table shows these values:

x	2.1	2.01	2.001	2.0001	2.00001	2.000001
f(x)	10	100	1000	10000	100000	1000000



When we find the right limit then it be increasing without maximum limit which can be written as : $\lim_{x\to 2^+}\frac{1}{x-2}=\infty$.

The symbol ∞ means that the limit is increasing without maximum limit. , Similarly it decreasing without minimum limit when we find the left limit which can be written as $\lim_{x\to 2^-}\frac{1}{x-2}=-\infty$

NOTE

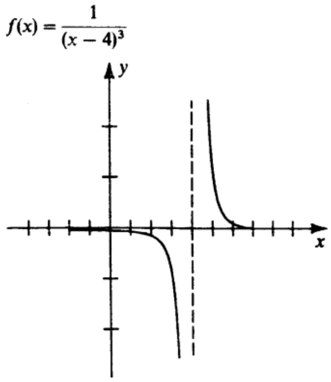
* If $\lim_{x\to 2}\frac{1}{x-2}=\infty$ then . $\lim_{x\to 2^-}\frac{1}{x-2}=\infty$ ، $\lim_{x\to 2^+}\frac{1}{x-2}=\infty$
* If $\lim_{x\to 2}\frac{1}{x-2}=-\infty$ then $\lim_{x\to 2^-}\frac{1}{x-2}=-\infty$ ، $\lim_{x\to 2^+}\frac{1}{x-2}=-\infty$

***Example* : 10**

Evaluate the following limits (if exist)

(i) $\lim\limits_{x\to4^-}\dfrac{1}{(x-4)^3}$, (ii) $\lim\limits_{x\to4^+}\dfrac{1}{(x-4)^3}$, (iii) $\lim\limits_{x\to4}\dfrac{1}{(x-4)^3}$

***Solution* :**



(i) If $x\to4,\ x<4$ then $\lim\limits_{x\to4^-}\dfrac{1}{(x-4)^3}=-\infty$.§§§ .

(ii) If $x\to4$, $x>4$ then $\lim\limits_{x\to4^+}\dfrac{1}{(x-4)^3}=\infty$.§§§.

(iii) As $\lim\limits_{x\to4^-}\dfrac{1}{(x-4)^3}\neq\lim\limits_{x\to4^+}\dfrac{1}{(x-4)^3}$, then

$\lim\limits_{x\to4}\dfrac{1}{(x-4)^3}$ doesn't exist .§§§

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***Theorem* : (3-7)**

If k is a positive rational number and c is a real number then:

1- $\lim\limits_{x\to-\infty}\dfrac{c}{x^k}=0$. $\lim\limits_{x\to\infty}\dfrac{c}{x^k}=0$ ‘

2- If n is an integer even number then: $\lim\limits_{x\to a}\dfrac{c}{(x-a)^n}=\infty$.

3- If n is an integer odd number then:

. $\lim\limits_{x\to a^+}\dfrac{c}{(x-a)^n}=\infty$ $\lim\limits_{x\to a^-}\dfrac{c}{(x-a)^n}=-\infty$ ‘

4- $\lim\limits_{x\to\infty}N^x=\infty$; $N>1$ ‘ $\lim\limits_{x\to\infty}N^x=0$; $N<1$

***Theorem* : (3-8)**

If $f(x), g(x)$ two functions defined on open interval doesn't contain the number a and $\lim\limits_{x\to a}f(x)=0$, $g(x)$ is bounded then: $\lim\limits_{x\to a}f(x)g(x)=0$

***Example* : 11**

Prove that: $\lim\limits_{x\to\infty} \dfrac{\cos x}{1+x^2} = 0$

***Solution* :**

Put $x = \infty$ we get : , $\lim\limits_{x\to\infty} \dfrac{\cos x}{1+x^2} = \dfrac{\infty}{\infty}$

Since $-1 \le \cos x \le 1$ then the function is bounded.

And $\lim\limits_{x\to\infty} \dfrac{1}{1+x^2} = 0$,

using theorem (3-8) then, $\lim\limits_{x\to\infty} \dfrac{1}{1+x^2}.\cos x = 0$. §§§

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***Example* : 12**

Prove that: $\lim\limits_{x\to\infty} \dfrac{2^{x+1} + 3.7^{x-1}}{7^x - 1} = \dfrac{3}{7}$

***Solution* :**

Put $x = \infty$, we get: $\lim\limits_{x\to\infty} \dfrac{2^{x+1} + 3.7^{x-1}}{7^x - 1} = \dfrac{\infty}{\infty}$

Divided by 7^x we get:

$$\lim_{x\to\infty} \frac{2^{x+1} + 3.7^{x-1}}{7^x - 1} \Rightarrow = \frac{2.0 + 3.7^{-1}}{1-0} = (3/7) \quad \lim_{x\to\infty} \frac{2.(2/7)^x + 3.7^{-1}}{1-(1/7)^x}$$ . . §§§

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***Example* : 13**

Evaluate : $\lim\limits_{x\to\infty} \dfrac{3x^3 - x + 1}{6x^3 + 2x^2 - 7}$

***Solution* :**

Put $x = \infty$ we get : $\lim\limits_{x\to\infty} \dfrac{3x^3 - x + 1}{6x^3 + 2x^2 - 7} = \dfrac{\infty}{\infty}$

Divided by x^3 we get :

$$\lim_{x\to\infty} \frac{3x^3 - x + 1}{6x^3 + 2x^2 - 7} = \lim_{x\to\infty} \frac{\cancel{x^3}\left(3 - (1/x^2) + (1/x^3)\right)}{\cancel{x^3}\left(6 + (1/x) - 7((1/x^3)\right)}$$

$$= \frac{(3 - 0 + 0)}{(6 + 0 - 0)} = \frac{1}{2}$$. §§§

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***Example* : 14**

Evaluate : $\lim\limits_{x\to\infty} \dfrac{\sqrt[5]{x^5 + x^3}}{\sqrt[7]{x^7 - x^2}}$

***Solution* :**

Put $x = \infty$, we get : $\lim\limits_{x\to\infty} \dfrac{\sqrt[5]{x^5 + x^3}}{\sqrt[7]{x^7 - x^2}} = \dfrac{\infty}{\infty}$

$$\lim_{x\to\infty} \frac{\sqrt[5]{x^5 + x^3}}{\sqrt[7]{x^7 - x^2}} = \lim_{x\to\infty} \frac{\sqrt[5]{x^5(1 + (1/x^2)}}{\sqrt[7]{x^7(1 - (1/x^2)}}$$

$$= \lim_{x\to\infty} \frac{\sqrt[5]{x^5(1 + (1/x^2)}}{\sqrt[7]{x^7(1 - (1/x^2)}} = \lim_{x\to\infty} \frac{x \ \sqrt[5]{(1 + (1/x^2)}}{x \ \sqrt[7]{(1 - (1/x^2)}} = \frac{1}{1} = 1$$

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I-5 Limits Of Trigonometric Functions

In this section we will have some theorems (without proof) which will help to find the limits of the functions which contains trigonometric functions.

***Theorem* : (3-9)**

(1) $\lim_{x\to 0} \sin x = 0$, (2) $\lim_{x\to 0} \cos x = 1$, (2) $\lim_{x\to 0} \tan x = 0$

***Theorem* : (3-10)**

(1) $\lim_{x\to a} \sin x = \sin a$, $\lim_{x\to a} \cos x = \cos a$, $\lim_{x\to a} \tan x = \tan a$; $a \in R$

(2) $\lim_{x\to 0} \frac{\sin x}{x} = 1$, $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$, $\lim_{x\to 0} \frac{\tan x}{x} = 1$

(3) $\lim_{x\to 0} \frac{\sin ax}{ax} = 1$, $\lim_{\theta\to 0} \frac{\tan ax}{ax} = 1$

***Example* : 15**

Evaluate : $\lim_{x\to 0} \frac{\sin 5x}{4x}$ (if exist)

***Solution* :**

Put $x = 0$, we get : $\lim_{x\to 0} \frac{\sin 5x}{4x} = \frac{0}{0}$,

Use theorem (3-10)

$$\lim_{x\to 0} \frac{\sin 5x}{4x} = \lim_{x\to 0} \frac{5}{4} . \frac{\sin 5x}{5x} = \frac{5}{4} . \lim_{x\to 0} \frac{\sin 5x}{5x} = \frac{5}{4} .1 = \frac{5}{4}$$. §§§

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***Example* : 16**

Evaluate : $\lim\limits_{\theta\to 0}\dfrac{1-\cos\theta}{\theta\sin\theta}$ (if exist)

***Solution* :**

Put $\theta=0$, we get : $\lim\limits_{\theta\to 0}\dfrac{1-\cos\theta}{\theta\sin\theta}=\dfrac{0}{0}$

Multiply the numerator and the denominator by the conjugate of $(1-\cos\theta)$ and using the theorem we get:

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta\sin\theta}=\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta\sin\theta}.\frac{1+\cos\theta}{1+\cos\theta}$$

$$\Rightarrow \lim_{\theta\to 0}\frac{1-\cos^2\theta}{(\theta\sin\theta)(1+\cos\theta)}=\lim_{\theta\to 0}\frac{\sin^2\theta}{(\theta\sin\theta)(1+\cos\theta)}$$

$$\Rightarrow \lim_{\theta\to 0}\frac{\sin\theta}{\theta}.\lim_{\theta\to 0}\frac{1}{1+\cos\theta}=1.\frac{1}{2}=\frac{1}{2}$$. §§§

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***Example* : 17**

Evaluate: $\lim\limits_{x\to 0}\dfrac{\sin 3x}{x+\tan 2x}$ (if exist)

***Solution* :**

Put $x=0$, we get :we get: x d by ivided , $\lim\limits_{x\to 0}\dfrac{\sin 3x}{x+\tan 2x}=\dfrac{0}{0}$

$$\lim_{x\to 0}\frac{\sin 3x}{x+\tan 2x}=\lim_{x\to 0}\frac{3.\dfrac{\sin 3x}{3x}}{\dfrac{x}{x}+2.\dfrac{\tan 2x}{2x}}$$

$$\Rightarrow \frac{3.\lim\limits_{x\to 0}\dfrac{\sin 3x}{3x}}{\lim\limits_{x\to 0}(1+2.\dfrac{\tan 2x}{2x})}=\frac{3.1}{1+2.1}=1.$$ §§§

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***Example* : 18**

Evaluate:_ $\lim\limits_{x\to\infty}x^2\sin\dfrac{1}{x}\tan\dfrac{2}{x}=2$ (if exist)

***Solution* :**

Put $y=\dfrac{1}{x}$ and noted that $x\to\infty\Leftrightarrow y\to 0$ then ,

$$\lim_{x\to\infty}x^2\sin\frac{1}{x}\tan\frac{2}{x}\Rightarrow\lim_{y\to 0}\frac{1}{y^2}.\sin y.\tan 2y$$

$$\Rightarrow \lim_{y\to 0}\frac{\sin y}{y}.2.\frac{\tan 2y}{2y}=1.2.1=2$$ §§§

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***Example* : 19**

Evaluate : (if exist) $\lim\limits_{x \to 0} \dfrac{\sin x + \tan x}{\sqrt{4x^2 - x^3}}$

***Solution* :**

Put $x = 0$ we get: $\lim\limits_{x \to 0} \dfrac{\sin x + \tan x}{\sqrt{4x^2 - x^3}} = \dfrac{0}{0}$

Divided by x we get:

$$\lim_{x \to 0} \frac{\dfrac{\sin x}{x} + \dfrac{\tan x}{x}}{\sqrt{\dfrac{4x^2 - x^3}{x^2}}} = \frac{\lim\limits_{x \to 0} \dfrac{\sin x}{x} + \lim\limits_{x \to 0} \dfrac{\tan x}{x}}{\sqrt{\lim\limits_{x \to 0} (4 - x)}} = \frac{1+1}{\sqrt{4}} = 1$$

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Exercise (3-3)

(1) Evaluate the following limits (if exist)

(a) $\lim\limits_{x \to 2} \dfrac{\sqrt{x}-\sqrt{2}}{x^2-2x}$, (b) . $\lim\limits_{x \to 1} \dfrac{|2-x|-1}{x^2-1}$

(c) $\lim\limits_{x \to -1} \dfrac{(x+3)^3-8}{x^2-7x-8}$, (d) . $\lim\limits_{x \to -2} \dfrac{x^3+8}{x^4-16}$

(e) $\lim\limits_{x \to (1/2)} \dfrac{16x^4-1}{2x+1}$, (f) . $\lim\limits_{x \to -\sqrt{3}} \dfrac{x^4-9}{x^5+9\sqrt{3}}$

(g) $\lim\limits_{x \to 0} \dfrac{(1+3x)^{10}-1}{(1+3x)^7-1}$, (h) $\lim\limits_{x \to 3} \sqrt[3]{\dfrac{2+5x-3x^3}{x^2-1}}$.

(2) Evaluate the following limits :

(a) $\lim\limits_{x \to -\infty} \dfrac{2x^2-3}{4x^3+5x}$, (b) $\lim\limits_{x \to \infty} \sqrt[3]{\dfrac{8+x^2}{x(x+1)}}$

(c) $\lim\limits_{x \to -\infty} \dfrac{4x-3}{\sqrt{x^2+1}}$, (d) $\lim\limits_{x \to -\infty} \dfrac{2x^2-3}{4x^3+5x}$

(3) Prove that:

(a) $\lim\limits_{x \to o^+} x^3 \sin \dfrac{1}{\sqrt{x}} = 0$, (b) $\lim\limits_{x \to \infty} x \sin \dfrac{1}{x} = 1$

(4) Evaluate the following limits (if exist)

(a) $\lim\limits_{x \to 0} \dfrac{\sin^2 2x}{x \tan 4x}$, (b) $\lim\limits_{x \to 0} \dfrac{\tan x - \sin x}{x^3}$

(c) $\lim\limits_{x \to 0} \dfrac{x \sin x}{1-\cos x}$, (d) $\lim\limits_{x \to 0} \sin 3x \cot 2x$

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II- CONTINUITY OF A FUNCTIONS

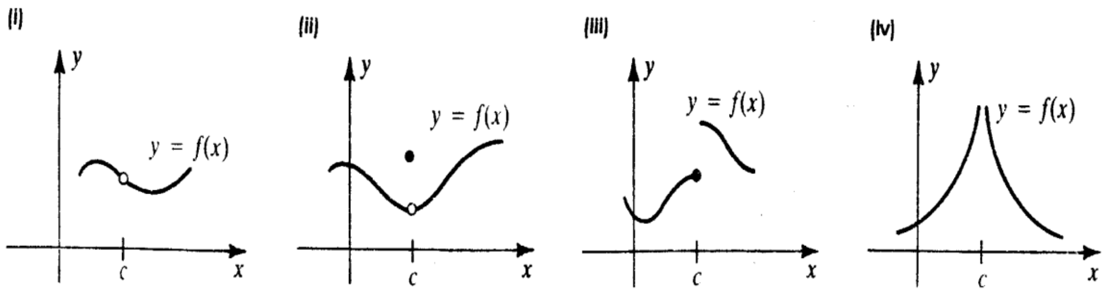

Fig. (i) *Fig. (i)*

From the figure:

- In figure (i) $f(c)$ is undefined.
- In figure (ii) $f(c)$ is undefined but $\lim_{x\to c} f(x) \neq f(c)$
- In figure (iii) $\lim_{x\to c} f(x)$ doesn't exist.
- In figure (iv) $f(c)$ is undefined and $\lim_{x\to c} f(x) = \infty$

Then the function $f(x)$ which is defined on open interval contain the point a is said to be continuous if it satisfies the conditions in the following definition.

Definition **: (3-2)**

The function $f(x)$ is said to be continuous at a if it satisfies the following conditions:

1- $f(x)$ is defined at $x = a$ ($f(a)$ is defined).

2- $f(x)$ has a limit when $x \to a$ ($\lim_{x\to a} f(x)$ is exist).

3- $\lim_{x\to a} f(x) = f(a)$.

N O T E

If at least one of the conditions in definition (3-5) not satisfied then the function is discontinuous.

Types of discontinuity:

** In figure (i),(ii) the discontinuity called **Removable Discontinuity .**

** In figure (iii) the discontinuity called **Jump Discontinuity**

** In figure (iv) the discontinuity called **Infinity Discontinuity.**

In the case of **Removable** we can redefined the function to be continuous.

***Example* : 20**

Show that the function $f(x)=\frac{x^2-1}{x-1}$; $x\neq 1$ is not continuous, and redefined it to be continuous.

***Solution* :**

From definition (3-2): $\lim_{x\to 1}\frac{x^2-1}{x-1}=2$ is not exist and $f(1)$

Then, the function can be written as following to be continuous:

$$f(x)=\begin{cases}\frac{x^2-1}{x-1} & ; \; x\neq 1\\ 2 & ; \; x=1\end{cases}$$

Then . $f(1)=2=\lim_{x\to 1} f(x)$ which is a continuous function §§§

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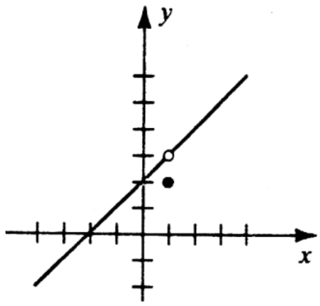
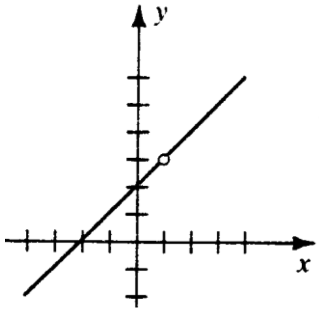
The following figures give some examples for the types of discontinuity.

i) $g(x)=\dfrac{x^2+x-2}{x-1}$, ii) $h(x)=\begin{cases}\dfrac{x^2+x-2}{x-1} & ; \ x\neq 1\\ 2\end{cases}$

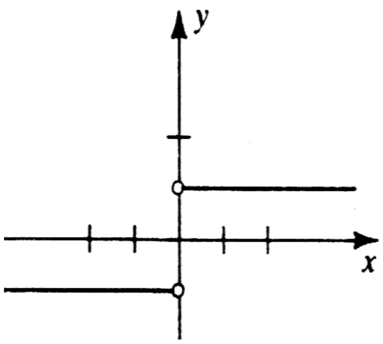


Removable discontinuity
defined g(1)

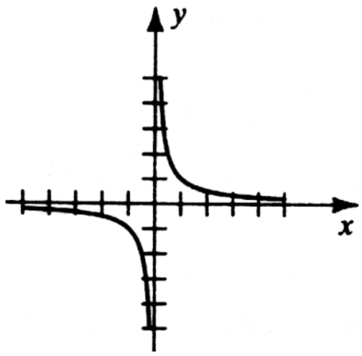


Removable discontinuity
$\lim_{x\to 1} h(x) = 3 \neq h(1)$

iv) $p(x)=\dfrac{|x|}{x}$ **iii)** $h(x)=\dfrac{1}{x}$



Jump Discontinuity
is un defined h(0)
doesn't exist $\lim_{x\to 0} h(x)$



Infinity Discontinuity
is un defined p(0)
doesn't exist $\lim_{x\to 0} p(x)$

***Theorem* : (3-11)**

1- The polynomial $f(x)$ is continuous for every real number $a\ (\forall a \in R)$.
2- The rational function $q = f / g$ is continuous for every real number except a where $g(a) = 0$.
3- The root function $h(x) = \sqrt{f(x)}$ is continuous for every real number except a where $f(a) < 0$.

***Example* : 21**

Find the points of discontinuity for the function: $f(x) = \dfrac{x^2 - 1}{x^3 + x^2 - 2x}$.

***Solution* :**

$$x^3 + x^2 - 2x = x(x^2 + x - 2) = x(x+2)(x-1)$$

Then the points of discontinuity is $0, -2, 1$. §§§

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***Example* : 22**

If $f(x) = \sqrt{9 - x^2}$, sketch the graph of f and show that the function is continuous on the interval $[-3,3]$.

***Solution* :**

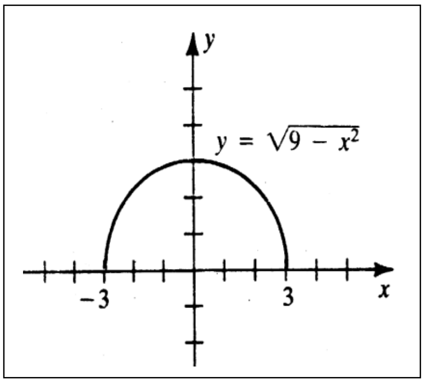
The continuity will be discussed firstly on the open interval (a,b) then we will discuss the continuity at a from the right and the continuity at b from the left.

Since $-3 < a < 3$ $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} \sqrt{9 - a^2} = f(a) \Rightarrow$

Then f is continuous for a on the open interval $(-3,3)$

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \sqrt{9 - x^2} = \sqrt{9-9} = 0 = f(-3)$$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \sqrt{9 - x^2} = \sqrt{9-9} = 0 = f(3)$$



Then , the function is continuous from the right at -3 and continuous from the left at 3 then, f is continuous on the closed interval $[-3,3]$. §§§

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***Example* : 23**

Discuss the continuity for the function: $f(x)=\begin{cases} x^2 & ; \ x\ge 0 \\ 1 & ; \ x<0 \end{cases}$

***Solution* :**

The continuity will be discussed at $x=0$,

1- $f(0)=0$ (Condition 1 satisfied).

2- $\lim\limits_{x\to 0^+} x^2 = 0 \ne 1 = \lim\limits_{x\to 0^-} 1$ then the limit doesn't exist(Condition 2 not satisfied).

then, the function is not continuous at $x=0$. §§§

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***Example* : 24**

Discuss the continuity for the following functions:

(a) $f(x)=\begin{cases} \sin x & ; \ x>0 \\ 1 & ; \ x=0 \\ x^2+x & ; \ 0>x>-1 \\ 0 & ; \ x\le -1 \end{cases}$, (b) $f(x)=\begin{cases} x^2+3x & ; \ x<0 \\ 5 & ; \ 0\le x<2 \\ \frac{x}{2} & ; \ 2\le x<3 \\ \frac{1}{2}\sqrt{3x} & ; \ x\ge 3 \end{cases}$

***Solution* :**

(a) we will discuss the continuity at the points $x=0$, $x=-1$.

First : at $x=0$

(1) $f(0)=1$ (Condition 1 satisfied).

(2) $\lim\limits_{x\to 0^+} \sin x = 0 = \lim\limits_{x\to 0^-} (x^2+x)$, then, $\lim\limits_{x\to 0} f(x)=0$ (Condition 2 satisfied).

(3) $\lim\limits_{x\to 0} f(x) = 0 \ne 1 = f(0)$ (Condition3 not satisfied)

then, the function is not continuous at $x=0$. . §§§

Second : at $x=-1$

(1) $f(-1)=0$ (Condition 1 satisfied).

(2) $\lim\limits_{x\to -1^+} (x^2+x) = 0 = \lim\limits_{x\to -1^-} 0$, then, $\lim\limits_{x\to 1} f(x)=0$ (Condition 2 satisfied).

(3) $\lim\limits_{x\to -1} f(x) = 0 = f(-1)$ (Condition 3 satisfied) , then, the function is continuous at $x=-1$. §§§

(b) we will discuss the continuity at the points $x=0$, $x=2$, $x=3$.

First : at $x=0$,

(1) $f(0)=5$ (Condition 1 satisfied).

(2) $\lim_{x\to 0^+} 5 = 5 \neq 0 = \lim_{x\to 0^-}(x^2+3x)$ then, $\lim_{x\to 0} f(x)$ doesn't exist. (Condition 2 not satisfied).

then, the function is not continuous at $x=0$.. §§§

Second : at $x=2$,

(1) $f(2)=(x/2)=1$, (Condition 1 satisfied).

(2) $\lim_{x\to 2^+}(x/2)=1\neq 5=\lim_{x\to 2^-} 5$ then, $\lim_{x\to 2} f(x)$ doesn't exist. (Condition 2 not satisfied).

then, the function is not continuous at $x=2$.. §§§

Finally : at $x=3$,

(1) $f(3)=\frac{1}{2}\sqrt{3x}=\frac{3}{2}$ (Condition 1 satisfied).

(2) $\lim_{x\to 3^+}\frac{1}{2}\sqrt{3x}=\frac{3}{2}=\lim_{x\to 3^-}(x/2)$ then, $\lim_{x\to 3} f(x)=\frac{3}{2}$ (Condition 2 satisfied).

(3) $\lim_{x\to 3} f(x)=\frac{3}{2}=f(3)$ (Condition 3 satisfied)

then, the function is continuous at $x=3$.. §§§

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T r y t o s o l v e :

Discuss the continuity for the following function:

(a) $f(x)=\begin{cases} |x+1|-1 & ; \ x<0 \\ \cos 2x & ; \ \geq 0 \end{cases}$, (b) $f(x)=\begin{cases} \sin x & ; \ x\leq 0 \\ \tan x & ; \ 0<x<\frac{\pi}{2} \\ 2x & ; \ x\geq \frac{\pi}{2} \end{cases}$

(c) $f(x)=|2x-5|$, (d) $f(x)=\begin{cases} \frac{\sin 2x}{x} & ; \ x>0 \\ 2 & ; x=0 \\ \frac{2x+x^2}{x} & ; \ x<0 \end{cases}$

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***Example* : 25**

Find the value of k which make the following functions be continuous at $x=0$.

(a) $f(x)=\begin{cases}\dfrac{\sin 4x}{3x} & ;x\neq 0\\ k & ;x=0\end{cases}$, (b) $f(x)=\begin{cases}\dfrac{\tan kx}{x} & ;x\neq 0\ ;\ \cos kx\neq 0\\ 5k-2 & ;x=0\end{cases}$

***Solution* :**

(a) $\lim\limits_{x\to 0}\dfrac{\sin 4x}{3x}=\lim\limits_{x\to 0}\dfrac{4}{3}\dfrac{\sin 4x}{4x}=\dfrac{4}{3}$

The function is continuous if the third condition satisfied then , $\lim\limits_{x\to 0}f(x)=f(0)\ \Rightarrow\ \dfrac{4}{3}=k$. §§§

(b) $\lim\limits_{x\to 0}\dfrac{\tan kx}{x}=\lim\limits_{x\to 0}k.\dfrac{\tan kx}{kx}=k$

The function is continuous if the third condition satisfied then, $.k=(1/2)\Rightarrow\ \lim\limits_{x\to 0}f(x)=f(0)\ \Rightarrow\ k=5k-2$. §§§

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***Example* : 26**

Find the values of k, L which make the following functions be continuous at $x=0$.

(a) $f(x)=\begin{cases}\dfrac{x^2-x+1}{3x^2+6} & ;x<0\\ L & ;\ x=0\\ \dfrac{x^2+\sin kx}{x} & ;\ x>0\end{cases}$, (b) $f(x)=\begin{cases}\sqrt{\dfrac{x+1}{x+4}} & ;x>0\\ k+1 & ;x=0\\ \dfrac{\sin Lx}{x} & ;x<0\end{cases}$

***Solution* :**

(a) $\lim\limits_{x\to 0^+}\dfrac{x^2+\sin kx}{x}=\lim\limits_{x\to 0^+}x+\lim\limits_{x\to 0^+}k.\dfrac{\sin kx}{kx}=0+k=k$

$\lim\limits_{x\to 0^-}\dfrac{x^2-6x+1}{3x^2+6}=\dfrac{1}{6}$.

Since, the function is continuous ,then

$$\lim_{x\to 0^-}f(x)=\frac{1}{6}=k=\lim_{x\to 0^+}f(x)\ \Rightarrow\ \lim_{x\to 0}f(x)=\frac{1}{6}$$

The function is continuous if the third condition satisfied then,

$$\lim_{x\to 0}f(x)=f(0)\ \Rightarrow\ k=\frac{1}{6}=L.$$

(b) $\lim_{x\to 0^+} \sqrt{\frac{x+1}{x+4}} = \frac{1}{2}$, $\lim_{x\to 0^-} \frac{\sin Lx}{x} = \lim_{x\to 0^-} L.\frac{\sin Lx}{Lx} = L$

$$\lim_{x\to 0^+} f(x) = \frac{1}{2} = L = \lim_{x\to 0^-} f(x) . \lim_{x\to 0} f(x) = \frac{1}{2} \Rightarrow$$

The function is continuous if the third condition satisfied then,

$$\lim_{x\to 0} f(x) = f(0) \Rightarrow \frac{1}{2} = L = k+1 \quad L = k = (1/2) \Rightarrow$$

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Try to solve:

Find the values of k, L which make the following functions be continuous

(a) $f(x) = \begin{cases} -5\sin x & ; x \le -\frac{\pi}{2} \\ L\sin x + k & ; -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 3+\cos x & ; x \ge \frac{\pi}{2} \end{cases}$, (b) $f(x) = \begin{cases} k - x & ; x < 4 \\ 1+2L & ; x = 4 \\ \sqrt{x} & ; x > 4 \end{cases}$

(c) $f(x) = \begin{cases} x^2 & ; x > 1 \\ L & ; x = 1 \\ (k^2+1)x - k & ; x < 1 \end{cases}$, (d) $f(x) = \begin{cases} \sqrt{\frac{x + L^2x^3}{x}} & ; x > 0 \\ x^2 - L & ; x < 0 \\ 2k & ; x = 0 \end{cases}$

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Exercise (3-4)

(1) Find the points of discontinuity for the following functions:

(a) $f(x)=\dfrac{x-1}{x^2+x-2}$, (b) . $f(x)=\dfrac{x-4}{x^2-x-12}$

(c) $f(x)=\dfrac{4x-7}{(x+3)(x^2+2x-8)}$, (d) $f(x)=\sqrt{2x-3}+x^2$

(2) Discus why the following functions are discontinuous and redefined it to be continuous (if possible)

(a) $f(x)=\begin{cases}\dfrac{x^2-9}{x-3} & ;\ x\neq 3\\ 4 & ;\ x=3\end{cases}$, (b) $f(x)=\begin{cases}\dfrac{\sin x}{x} & ;\ x\neq 0\\ 0 & ;\ x=0\end{cases}$

(c) $f(x)=\begin{cases}|x+3| & ;\ x\neq -3\\ 2 & ;\ x=-3\end{cases}$, (d) $f(x)=\begin{cases}\dfrac{1-\cos x}{x} & ;\ x\neq 0\\ 1 & ;\ x=0\end{cases}$

(3) Discus the continuity for the following functions:

(a) $f(x)=\begin{cases}\dfrac{\sqrt{3+x}-\sqrt{3}}{x-3} & ;\ x\neq 0\\ \dfrac{1}{2\sqrt{3}} & ;\ x=0\end{cases}$, (b) $f(x)=\begin{cases}\dfrac{2x-2}{|x-1|} & ;\ x\neq 1\\ 0 & ;\ x=1\end{cases}$

(c) $f(x)=\begin{cases}\dfrac{\sqrt{2x+3}-\sqrt{x+2}}{x-3} & ;\ x>-1\\ \dfrac{x^2-2}{x^2-x-2} & ;\ x<-1\end{cases}$, (d) $f(x)=\begin{cases}\dfrac{2x-2}{|x-1|} & ;\ x\neq 1\\ 0 & ;\ x=1\end{cases}$

(4) Find the values of k, L which make the following functions be continuous

(a) $f(x)=\begin{cases}\dfrac{\sqrt{6x-5}-\sqrt{3x+10}}{x+1} & ;\ x>5\\ \dfrac{L}{10} & ;\ x=5\\ \dfrac{7|x-5|}{x^3-125} & ;\ x<5\end{cases}$, (b) $f(x)=\begin{cases}\dfrac{x^2-4}{|x-2|} & ;\ x<2\\ L & ;\ x=2\\ kx & ;\ x>2\end{cases}$

(c) $f(x)=\begin{cases}\dfrac{\tan^2 x+3x\sin x}{x^2} & ;\ x\neq 0\\ k & ;\ x=0\end{cases}$, (d) $f(x)=\begin{cases}\dfrac{x\sqrt{x^2+1}}{|x|} & ;\ x<0\\ 2x^2-3x+L & ;\ x\geq 0\end{cases}$

(e) $f(x)=\begin{cases}\dfrac{1-\sqrt{x}}{x-1} & ;\ 0\leq x<1\\ k & ;\ x=1\end{cases}$, (f) $f(x)=\begin{cases}x^3 & ;\ x<-1\\ kx+L & ;\ -1\leq x<1\\ x^2+2 & ;\ x\geq 1\end{cases}$

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CHAPTER 4

DERIVATIVES AND DERIVATIVES APPLICATIONS

I- DERIVATIVES

Derivatives are used to calculate velocity and acceleration , to estimate the rate of spread of a disease, to set levels of production so as to maximize efficiency, to find the best dimensions of a cylindrical can, to find the age of a prehistoric artifact, and for many other applications. In this chapter, we introduce techniques to calculate derivatives easily and learn how to use derivatives to approximate complicated functions.

Definition :* *(4.1)

The derivative of the function $f(x)$ with respect to the variable x is the function whose value at x is: $f^{/}(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$; Provided the limit exists

I-1 Derivatives By Definition

The process of calculating a derivative is called differentiation. To emphasize the idea that differentiation is an operation performed on a function we use the notation $\frac{d}{dx}f(x)$.

If we use the traditional notion $y=f(x)$ to indicate that the independent variable x and the dependent variable y then some common alternative notions for the derivative are as follows: $f'(x)=y'=\frac{dy}{dx}=\frac{df}{dx}=\frac{d}{dx}f(x)=Df(x)=D_x f(x)$

The symbols $\frac{d}{dx}$ and D indicate the operation of differentiation and are called differentiation operators. We read $\frac{dy}{dx}$ as "the derivative of y with respect to x", and $\frac{df}{dx}$, $y=x^{2/3}(8x^3-5x+4)$ as " the derivative of f with respect to x ". The "prime" notations y' and f' come from notations that Newton used for derivatives.

***Example* : 1**

By derivative definition , find $f^{/}(x)$ if . $f(x)=x^2$

Solution:

$$f(x)=x^2 \Rightarrow f(x+h)=(x+h)^2$$

$$f^{/}(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2-(x^2)}{h}$$

$$=\lim_{h\to 0}\frac{2xh+h^2}{h}=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$

§§§ $=\lim_{h\to 0} 2x+h=2x.$

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***Example* : 2**

By derivative definition , find $f^{/}(x)$ if . $f(x)=2x+3$

Solution:

$$f^{/}(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{2h}{h}=2 \;=\lim_{h\to 0}\frac{\big(2(x+h)+3\big)-\big(2x+3\big)}{h}$$. §§§

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***Example* : 3**

If $f(x)=3x^2-12x+8$ by using derivative definition then :

a) Find $f^{/}(-2)$, $f^{/}(x)$

b) The slope of the tangent to the curve at the point $P(3,-1)$.

c) The points at which the tangent of the curve is horizontal.

H I N T

The slope of the tangent to the curve at a point is equal to the first derivative of the function at this point.

Solution:

a) $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

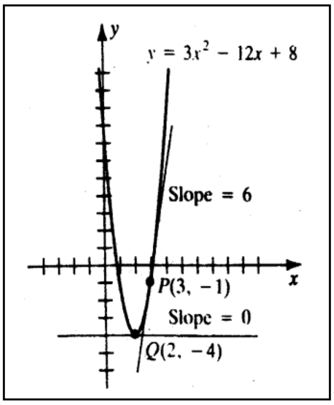
$$= \lim_{h\to 0}\frac{\left(3(x+h)^2 - 12(x+h)+8\right)-\left(3x^2-12x+8\right)}{h}$$

$$= \lim_{h\to 0}\frac{6xh+3h^2-12h}{h} = \lim_{h\to 0}(6x+3h-12)$$ §§§

$$\therefore\ f'(x) = (6x-12)$$. §§§

$$f'(-2) = 6(-2)-12 = -24$$ §§§

b) $$f'(x)\Big]_{(3,-1)} = (6x-12)\Big]_{(3,-1)} = 6(3)-12 = 6$$



c) Since the tangent is horizontal, then the slope equal to zero.

i.e. $f'(x)=0 \Rightarrow 6x-12=0 \Rightarrow x=2$ and $y=-4$

$\therefore$ The tangent is horizontal at the point $Q(2,-4)$, as in the following figure . §§§

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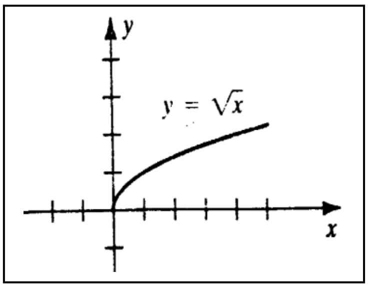
Example : 4

If $f(x)=\sqrt{x}$ sketch the graph of $f(x)$ and find $f'(x)$.

Solution:



$$f'(x) = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h\to 0}\frac{(x+h)-x}{h\,(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0}\frac{1}{(\sqrt{x+h}+\sqrt{x})} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$ §§§

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Example : 5

Prove that: $f(x)=\sqrt{x}$ is not differentiable at $x=0$.

Solution:

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+}\frac{\sqrt{0+h}-\sqrt{0}}{h}$$

$$= \lim_{h\to 0^+}\frac{\sqrt{h}}{h} = \lim_{h\to 0^+}\frac{1}{\sqrt{h}} = \infty$$

Since the limit is not finite, there is no derivative at $x=0$ .Then, the graph has a vertical tangent at the origin

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***Example* : 6**

Prove that the function $f(x) = |x|$ is not differentiable at $x = 0$

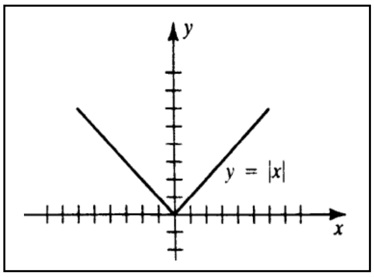
Solution:

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0^+} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = 1$$

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0^-} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = -1 \cdot$$



$\therefore$ the function $f(x) = |x|$ is not differentiable at $x = 0$

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Try To Solve

(1)For the following functions find:

a) The first derivative f'.

b) The equation of tangent for the curve at the point P.

c) The point Q at which the tangent is horizontal.

(i) $f(x) = 3x^2 - 2x - 4$; $P(2,4)$, (ii). $f(x) = x^3 - 4x - 4$; $P(2,0)$

(2) For the following functions: prove that the first derivative f' doesn't exist at the given Number a

(a) $f(x) = |x-5|$; $a = 5$, (b) $f(x) = |x+5|$; $a = -2$

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***Theorem* : (4-1)**

The function $f(x)$ is differentiable at $x = a$ if and only if it is continuous at a .

I-2 Derivatives By Rules

RULE : 1

** If $f'(x) = m$, then $f(x) = mx + b$, where m, b are constants.

** If $f(x) = b$, then $f'(x) = 0$, where *b* is constant

e.g.

$f(x)$	$3x-7$	$-4x+2$	$7x$	π^2	13
$f'(x)$	3	-4	7	0	0

RULE : 2 (Power Rule)

** If $n \in Q$ and $f(x) = x^n$; $x \neq 0$, then $f'(x) = nx^{n-1}$.

** If $f(x) = cx^n$, then $f'(x) = (cn)\,x^{n-1}$.

e.g.

$f(x)$	x^3	$5x^{10}$	$x^{-1} = \frac{1}{x}$	$x^{-2} = \frac{1}{x^2}$	$x^{-10} = \frac{1}{x^{10}}$
$f'(x)$	$3x^2$	$50x^9$	$-x^{-2} = -\frac{1}{x^2}$	$-2x^{-3} = -\frac{2}{x^3}$	$-10x^{-11} = -\frac{10}{x^{11}}$

e.g.

$f(x)$	$\sqrt{x} = x^{1/2}$	$\sqrt[3]{x^2} = x^{2/3}$	$\frac{1}{\sqrt[4]{x^5}} = x^{-5/4}$
$f'(x)$	$\frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$	$\frac{2}{3}x^{-1/3} = \frac{2}{3}\frac{1}{\sqrt[3]{x}}$	$\frac{-5}{4}x^{-9/4} = \frac{-5}{4}\frac{1}{\sqrt[4]{x^9}}$

e.g.

- ► $D_x(3x^7) = (3.7)x^6 = 21x^6$
- ► $D_t(5\,t^{12}) = (5.12)\,t^{11} = 60\,t^{11}$
- ► $\frac{d}{dx}(4x^{3/2}) = (4.\frac{3}{2})x^{1/2} = 6x^{1/2}$
- ► $\frac{d}{dr}(2\,r^{-4}) = 2(-4)r^{-5} = -8r^{-5} = \frac{-8}{r^5}$
- ► $D_x(x^3)\big]_{x=5} = 3x^2\big]_{x=5} = 3(5^2) = 75$
- ► $D_x(9x^{4/3})\big]_{x=8} = 9.(4/3)x^{1/3}\big]_{x=8} = 12(8^{1/3}) = 24$

I-3 Higher Order Derivatives

The Higher order derivatives derivative symbols

$f^{/}(x), f^{//}(x), f^{///}(x), f^{(4)}(x), ..., f^{(n)}(x)$ **

** $D_x\, y,\ D_x^2\, y,\ D_x^3\, y,\ D_x^4\, y, ...,\ D_x^n\, y$

** $y^{/}, y^{//}, y^{///}, y^{(4)}, ..., y^{(n)}$

** $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, ..., \frac{d^ny}{dx^n}$

Example **: 7**

Find $f^{(4)}(x)$ for the function $f(x) = 4x^{3/2}$

Solution:

$f(x) = 4x^{3/2}$, then

► . $f^{/}(x) = 4.(\frac{3}{2}).x^{1/2} = 6x^{1/2}$

► . $f^{//}(x) = 6.(\frac{1}{2})x^{-1/2} = 3x^{-1/2}$

► . $f^{(3)}(x) = 3(-\frac{1}{2})x^{-3/2} = -\frac{3}{2}x^{-3/2}$

► . $f^{(4)}(x) = -\frac{3}{2}.(-\frac{3}{2})x^{-5/2} = \frac{9}{4}x^{-5/2}$

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Theorem **: (4-2)**

let f, g are two differentiable functions, c, m, b are real numbers and n is a rational number then:

(1) $D_x\,[c\,f(x)] = c\ D_x f(x)$

(2) $D_x[f(x) \pm g(x)] = D_x f(x) \pm D_x g(x) = f^{/}(x) \pm g^{/}(x)$

(3) $D_x[f(x).g(x)] = f(x).g^{/}(x) + g(x).f^{/}(x)$

(4) $D_x\left[\frac{f(x)}{g(x)}\right] = \frac{g(x).f^{/}(x) - f(x).g^{/}(x)}{[g(x)]^2}$; $g(x) \neq 0$

***Example* : 8**

Find the first derivative for the following functions:

(a) $f(x) = 2x^4 - 5x^3 - 4$, (b) $h(x) = (x^3 + 1)(2x^2 + 8x)$, (c) $k(x) = \frac{3x^2 - 2x - 4}{4x^2 + 5}$.

Solution:

(a) $f(x) = 2x^4 - 5x^3 - 4 \quad \Rightarrow$

$. = 8x^3 - 15x^2 \quad f^{/}(x) = (2.4)\, x^3 - (5.3)x^2 \quad \therefore$ §§

(b) $h(x) = \underbrace{(x^3 + 1)}_{f}\ \underbrace{(2x^2 + 8x)}_{g} \quad \Rightarrow$

$h^{/}(x) = \underbrace{(x^3 + 1)}_{f}.\underbrace{[4x + 8]}_{g^{/}} + \underbrace{(2x^2 + 8x)}_{g}.\underbrace{[3x^2]}_{f^{/}}$.

$= 4(x^3 + 1)(x + 2) + 2x(x + 2).3x^2$

$\therefore \quad h^{/}(x) = 2(x + 2)(5x^3 + 2)$. §§

(c) $k(x) = \frac{\overbrace{3x^2 - 2x - 4}^{f}}{\underbrace{4x^2 + 5}_{g}} \quad \Rightarrow$

$$k^{/}(x) = \frac{\overbrace{(4x^2 + 5)}^{g}\ \overbrace{[6x - 2]}^{f^{/}} - \overbrace{(3x^2 - 2x - 4)}^{f}\ \overbrace{[8x]}^{g^{/}}}{\underbrace{[4x^2 + 5]^2}_{g^2}}$$

$$\therefore \quad k^{/}(x) = \frac{2(4x^2 + 5)(6x - 2) - 8x(3x^2 - 2x - 4)}{(4x^2 + 5)^2} .$$ §§§

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***Example* : 9**

Find an equation for the tangent to the curve:at the point (1, 3) $y = x + (2/x)$

Solution:

The slope of the curve is $\frac{dy}{dx} = 1 + 2(\frac{-1}{x^2})$.

The slope at $x = 1$ is $\left.\frac{dy}{dx}\right|_{x=1} = \left[1 - \frac{-2}{x^2}\right]_{x=1} = 1 - 2 = -1.$

Then, the equation of the tangent through the point (1, 3) with $\frac{y-3}{x-1} = -1$ is $m = -1$

$\Rightarrow \quad y = -x + 4 \quad \Rightarrow \quad y - 3 = (-1)(x - 1)$

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I-4 Derivatives Of A Trigonometric Functions

Many of the phenomena we want information about are approximately periodic (electromagnetic fields, heart rhythms, tides, weather). The derivatives of sines and cosines play a key role in describing periodic changes. This section shows how to differentiate the six basic trigonometric functions.

***RULE* : 3**

(1) $D_x[\sin x] = \cos x$, **(2)** $D_x[\cos x] = -\sin x$

(3) $D_x[\tan x] = \sec^2 x$, **(4)** $D_x[\cot x] = -\csc^2 x$

(5) $D_x[\sec x] = \sec x.\tan x$, **(6)** $D_x[\csc x] = -\text{cecx}.\cot x$

**** The above rules can be proved by using derivative definition as follows :**

1) $D_x[\sin x] = \cos x$

$$f(x) = \sin x, f(x+h) = \sin(x+h)$$

$$f^{/}(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x.\cosh + \cos x.\sinh - \sin x}{h}$$

$$= \lim_{h\to 0}\sin x\frac{\cosh - 1}{h} + \lim_{h\to 0}\cos x\frac{\sinh}{h}$$

§§§ $= \sin x.(0) + \cos x.(1) = \cos x.$

2) $D_x[\cos x] = -\sin x$

$$f(x) = \cos x, f(x+h) = \cos(x+h)$$

$$f^{/}(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$= \lim_{h\to 0}\frac{\cos x.\cosh - \sin x.\sinh - \cos x}{h}$$

$$= \lim_{h\to 0}\cos x.\frac{\cosh - 1}{h} - \lim_{h\to 0}\sin x\frac{\sinh}{h}$$

$= \cos x.(0) - \sin x.(1) = -\sin x.$ §§§

3) $D_x[\tan x] = \sec^2 x$

$$f(x) = \tan x = \frac{\sin x}{\cos x}$$

$$f^{/}(x) = \frac{\cos x.\cos x - \sin x.(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

§§§. $= \frac{1}{\cos^2 x} = \sec^2 x.$

4) $D_x[\cot x] = -\csc^2 x$

$$f(x) = \cot x = \frac{\cos x}{\sin x}$$

$$f^{/}(x) = \frac{\sin x.(-\sin x) - \cos x.(\cos x)}{\sin^2 x} = \frac{-(\cos^2 x + \sin^2 x)}{\sin^2 x}$$

$= \frac{-1}{\sin^2 x} = -\csc^2 x.$.§§§

5) $D_x[\sec x] = \sec x.\tan x$

$$f(x) = \sec x = \frac{1}{\cos x}$$

$$f^{/}(x) = \frac{\cos x.(0) - (-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}.\frac{\sin x}{\cos x}$$

§§§. $= \sec x.\tan x$

6) $D_x[\csc x] = -\csc x.\cot x$

$$f(x) = \csc x = \frac{1}{\sin x}$$

$$f^{/}(x) = \frac{\sin x.(0) - (\cos x)}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = \frac{-1}{\sin x}.\frac{\cos x}{\sin x}$$

§§§. $= -\csc x.\cot x.$

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***Example* : 10**

Find the first derivative $y^{/}$ for the following functions:

(a) $y = \sec x.\tan x$, (b) $y = 2x.\cot x + x^2 \tan x$, (c) $y = \dfrac{\sin x}{1+\cos x}$

Solution:

(a) as : $y = \sec x.\ \tan x \quad \Rightarrow$

$$y^{/} = \sec x.\left[\sec^2 x\right] + \tan x.\left[\sec x.\tan x\right]$$

$$= \sec x.\left[\sec^2 x + \tan^2 x\right].§§§$$

(b) as : $y = 2x.\cot x + x^2 \tan x \quad \Rightarrow$

$$y^{/} = \left(2x.(-\csc^2 x) + \cot x.(2)\right) + \left(x^2(\sec^2 x) + \tan x.(2x)\right)$$

$$y^{/} = 2(-x.\csc^2 x + \cot x) + x(x.\sec^2 x) + 2\tan x) \quad .§§§$$

(c) as : $y = \dfrac{\sin x}{1+\cos x} \quad \Rightarrow$

$$y^{/} = \frac{(1+\cos x)[\cos x] - \sin x[-\sin x]}{(1+\cos x)^2}$$

$$= \frac{1+(\cos^2 x + \sin^2 x)}{(1+\cos x)^2} = \frac{2}{(1+\cos x)^2} .§§§ \qquad \{ \text{ as : } \sin^2 x + \cos^2 x = 1 \}$$

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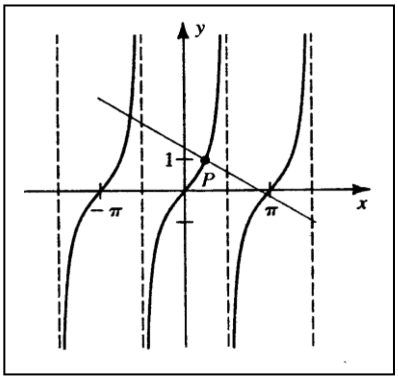
***Example* : 11**

Find the equation of normal line on the curve $y = \tan x$ at the point. $P(\frac{\pi}{4}, 1)$

Solution:



$$y^{/} = \sec^2 x = m_p$$

The slope of tangent: $m_P = \sec^2(\pi/4) = (\sqrt{2})^2 = 2$.

The slope of normal $m_{\perp} = -1/m_P = -1/2$

Then, the equation of normal is:

$$y - 1 = -\frac{1}{2}(x - \frac{\pi}{4}) \quad \Rightarrow \quad y = -\frac{1}{2}x + \frac{\pi}{8} + 1 \quad .§§§$$

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Exercise (4-1)

(1) Use the definition of the derivative to find the first derivative for the following functions

(a) $y = 5x^2 - 4x$, (b) $k(x) = 3 - 2x^2$

(2) Find the first derivative for the following functions

(a) $y = 5x^4 - \sqrt[4]{x^3}$, (b) $k(x) = x^{2/3}.(8x^3 - 5x + 4)$

(c) $h(x) = \dfrac{8x^2 - 6x + 11}{4x - 1}$, (d) $k(x) = \dfrac{x^2 - 3x}{\sqrt[3]{x^2}}$

(e) $y = (x + \csc x).\cot x$, (f) $y = (\sin x + \cos x)^2$

(g) $y = \dfrac{1}{\sin x.\tan x}$, (h) $y = 3x^2 \sec x - x^3.\tan x$

(i) $y = \sin(-x) + \cos(-x)$, (j) $y = \dfrac{1 + \sec(-x)}{\tan x + \sin x}$

{ Hint : $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$, $\tan(-x) = -\tan x$ }

(3) Find the equations of tangent and normal lines for the following curves at the pint *P*

(a) $f(x) = \dfrac{5}{1 + x^2}$; $P(-2,1)$, (b) $y = 3x^2 - 2\sqrt{x}$; $P(4, 44)$

(c) $f(x) = \sec x$; $P(\frac{\pi}{4}, f(\frac{\pi}{4}))$, (d) $f(x) = \csc x + \cot x$; $P(\frac{\pi}{4}, f(\frac{\pi}{4}))$

(e) $f(x) = x + 2\cos x$; $P(0,0)$, (f) $f(x) = x + \sin x$; $P(\frac{\pi}{2}, f(\frac{\pi}{2}))$

(4)Find $y^{//}$ for the following functions

(a) $y = 6x^4 + 2\cos x$, (b) $y = 16\sqrt[4]{x^3}$, (c) $y = \sec x$

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I-5 Chain Rule

We know how to differentiate $y = f(u) = \sin u$ and we , but how do $u = g(x) = x^2 - 4$ differentiate a composite like $F(x) = f(g(x)) = (\sin x^2 - 4)$? .

The differentiation formulas we have studied so far do not tell us how to calculate $f^{/}(x)$ So how do we find the derivative ofhain Rule, wer is, with the C? The ans $F = f \circ g$ which says that the derivative of the composite of two differentiable functions is the product of their derivatives evaluated at appropriate points. The Chain Rule is one of the most important and widely used rules of differentiation. This section describes the rule and how to use it.

e.g. *:*

as , then $\sin 2x = 2\text{six}.\cos x$

$$D_x \ \sin 2x = D_x(2.\underbrace{\sin x}_{f}.\underbrace{\cos x}_{g})$$

$$= 2.\underbrace{\sin x}_{f}.[\underbrace{(-\sin x)}_{g^{/}} + \underbrace{(\cos x)}_{f^{/}}.\underbrace{\cos x}_{g}]$$

$$= 2.\left(-\sin^2 x + \cos^2 x\right) = 2\cos 2x \quad . \quad \{ \text{ as} : \cos 2x = (\cos^2 x - \sin^2 x) \ \}$$

Then : $D_x \ \sin 2x = 2\cos 2x$.

** We can use the following note to evaluate $D_x \ \sin 2x$:

If : $y = \sin u = f(u)$ $u = 2x = g(x)$ ،

then, , and $y = \sin 2x = f(g(x)) = f \circ g$

$$y^{/} = \frac{dy}{du} = \cos u = f^{/}(u) \quad , \frac{du}{dx} = 2 = g^{/}(x) \quad , \quad \frac{dy}{dx} = (f \circ g)^{/}(x)$$

$$\therefore \qquad D_x(\sin 2x) = D_x f(g(x)) = \underbrace{\cos u}_{f^{/}} . \underbrace{2}_{g^{/}}$$

$$= 2\cos 2x \ .$$

Then we noted that: $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ which called chain method.

RULE : 4 (Chain rule)

If $y = f(u)$, $u = g(x)$ and $\frac{dy}{du}$ $\frac{du}{dx}$ ، exist then,

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = f^{/}(g(x)).g^{/}(x)$$

***Example* : 12**

Find $y^{/} = \frac{dy}{dx}$ for the following functions

$y = \sqrt{u}$

(a) $x = a(u - \sin u)$; $y = a(u - \cos u)$

(b) $u = x^2 + 1$

Solution:

(a) $\Rightarrow$ $x = a(u - \sin u)$; $y = a(u - \cos u)$

Using the chain rule: $y^{/} = \frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$

$$\frac{dx}{du} = a(1 - \cos u) \quad , \quad \frac{dy}{du} = a(1 + \sin u) \quad \Rightarrow$$

$$y^{/} = \frac{dy}{du} . \frac{du}{dx} = a(1 + \sin u) . \frac{1}{a(1 - \cos u)}$$

$$\therefore \quad y^{/} = \frac{(1 + \sin u)}{(1 - \cos u)} \quad .$$ §§§

(b) If we put : we get y in $u = x^2 + 1$

$$y = \sqrt{u} = \sqrt{x^2 + 1} = (x^2 + 1)^{1/2} \qquad \therefore \quad y^{/} = \frac{x}{\sqrt{x^2 + 1}}$$

Or, by using chain rule

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx} = \left(\frac{1}{2} u^{-1/2}\right) . (2x)$$

$$\therefore \quad y^{/} = \frac{x}{\sqrt{u}} = \frac{x}{\sqrt{x^2 + 1}} \quad .$$ §§§

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***R ULE* : 5**

If $n \in Q$, $y = u^n$, $u = g(x)$ rational number) then :

$$D_x(u^n) = n\, u^{n-1} . D_x u$$

which is equivalent to the following relation:

$$D_x[g(x)]^n = n.[g(x)]^{n-1} .\ g^{/}(x)$$

Example **: 13**

Find $y^{/} = \frac{dy}{dx}$ for the following functions :

(a) $y = \frac{1}{(4x^2 + 6x - 7)^3}$, (b) $y = (3x+1)^6 . \sqrt[3]{2x-5}$, (c) $y = \cos^3 x . \sqrt{\sec x}$

Solution:

(a) $y = \frac{1}{(4x^2 + 6x - 7)^3} \Rightarrow y = (4x^2 + 6x - 7)^{-3}$

$\therefore\ y^{/} = -3.(4x^2 + 6x - 7)^{-4}.(8x + 6)$. §§§

(b) $y = (3x+1)^6.(2x-5)^{1/3} \Rightarrow y = (3x+1)^6 . \sqrt[3]{2x-5}$

If : $f(x) = (3x+1)^6$, $h(x) = (2x-5)^{1/3}$

$\therefore\ y^{/} = (3x+1)^6 . \left[\frac{1}{3}(2x-5)^{-2/3}(2)\right] + \left[6(3x+1)^5.(3)\right](2x-5)^{1/3}$. §§§

(c) $y = (\cos x)^3 . (\sec x)^{1/2} \Rightarrow y = \cos^3 x . \sqrt{\sec x}$

If : $f(x) = (\cos x)^3$, $h(x) = (\sec x)^{1/2}$

$$\therefore\ y^{/} = (\cos x)^3 \left(\frac{1}{2}(\sec x)^{-1/2}(\sec x . \tan x)\right) + (\sec x)^{1/2}\left(3(\cos x)^2(-\sin x)\right).$$ §§§

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Theorem **: (4-3)**

If $u = g(x)$ and g is differentiable then :

(1) $D_x[\sin u] = \cos u . D_x u$, **(2)** $D_x[\cos u] = -\sin u . D_x u$

(3) $D_x[\tan u] = \sec^2 u . D_x u$, **(4)** $D_x[\cot u] = -\csc^2 u . D_x u$

(5) $D_x[\sec u] = \sec u . \tan u . D_x u$, **(6)** $D_x[\csc u] = -\csc u . \cot u . D_x u$

***Example* : 14**

Find tionsng funcfor the followi $y^{/} = \frac{dy}{dx}$

(a) $y = \sin(x^2 + 3x)$, (b) $y = \tan(\sec x)$, (c) $y = \cos\sqrt{x + \csc 3x}$

Solution:

(a) $y^{/} = \cos(x^2 + 3x).[2x + 3]$ $\therefore \Rightarrow$ $y = \sin(x^2 + 3x)$. §§§

(b) $y = \tan(\sec x)$ $\Rightarrow \therefore$ $y^{/} = \sec^2(\sec x).[\sec x.\tan x]$. §§§

(c) $y = \cos\sqrt{x + \csc 3x}$

Let : $u = \sqrt{x + \csc 3x} = (x + \csc 3x)^{1/2}$

$$D_x u = \frac{1}{2}(x + \csc 3x)^{-1/2}.[1 - 3.\csc 3x.\cot 3x]$$

$$\therefore\ y^{/} = -\sin\sqrt{x + \csc 3x}.\left(\frac{1}{2}(x + \csc 3x)^{-1/2}.[1 - 3.\csc 3x.\cot 3x]\right)$$. §§§

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***Example* : 15**

For the curve, find the points at which the tangent is $y = \cos 2x + 2\cos x$; $0 \le x \le 2\pi$:

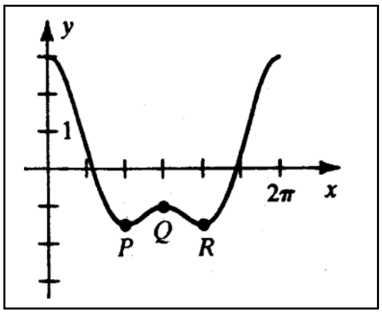
horizontal

Solution:

$= (-\sin 2x)D_x(2x) + 2(-\sin x)$ $D_x y = D_x(\cos 2x + 2\cos x)$

$$= -2\sin 2x - 2\sin x$$

The tangent is horizontal if ($D_x y = 0$)



$\sin 2x + \sin x = 0 \iff -2\sin 2x - 2\sin x = 0$

(as :) $\sin 2x = 2\sin x.\cos x$

$\sin 2x + \sin x = 2\sin x.\cos x + \sin x = 0$

$\Rightarrow$ $\sin x(2\cos x + 1) = 0$

$\Rightarrow$ $\sin x = 0 \ \vee \ (2\cos x + 1) = 0$

i) When $\sin x = 0$ then the values of x are 0 , π, 2π

ii) When $(2\cos x + 1) = 0 \iff \cos x = -\frac{1}{2}$

then the values of x are $\frac{2\pi}{3}$, $\frac{4\pi}{3}$

$$\therefore\ x \in \left\{0\ ,\ \frac{2\pi}{3}\ ,\pi, \frac{4\pi}{3}, 2\pi\right\}$$. §§§

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I-6 <u>Implicit Differentiation</u>

Most of the functions we have dealt with so far have been described by an equation of the form that expresses $y = f(x)$ explicitly in terms of the variable x. We have learned rules for differentiating functions defined in this way. Some situations occur when we encounter equations like: $y^2 - x = 0$, $x^2 + y^2 - 25 = 0$, . $x^3 + y^3 - 9xy = 0$

These equations define an implicit relation between the variables x and y. In some cases we may be able to solve such an equation for y as an explicit function (or even several functions) of x. When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate it in the usual way, we may still be able to find $\frac{dy}{dx}$ by implicit differentiation. This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for $y^\backslash$. This section describes the technique and uses it to extend the Power Rule for differentiation to include rational exponents.

***<u>Example</u>* : 16**

Find $y^/$ for the implicit function : $y^4 + 3y - 4x^3 = 5x + 1$

<u>Solution:</u>

$$4.y^3.y^/ + 3y^/ = (12x^2 + 5)$$

$$\Rightarrow \quad (4.y^{3^/} + 3)y^/ = (12x^2 + 5)$$

$$\therefore \quad y^/ = \frac{(12x^2 + 5)}{(4.y^{3^/} + 3)} \; .$$ §§§

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***<u>RULE</u>* : 6**

If $f(x, y) = 0$ is an implicit function then :

$$D_x(y^n) = n\ y^{n-1}.D_x y = ny^{n-1}y^/$$

Example : 17

Find $y^/$ for the following functions

(a) $4xy^3 - x^2y + x^3 - 5x + 6 = 0$, find the slope at the point $(1,-1)$,

(b) $y = x^2 \sin y$

Solution:

(a) $4xy^3 - x^2y + x^3 - 5x + 6 = 0$

$$\Rightarrow (4.y^3 + 4x.3y^2y^/) - (2x.y + x^2y^/) - 5x = 0$$

$$\Rightarrow (12xy^2 - x^2)\, y^/ = (-4y^3 + 2xy - 3x^2 + 5)$$

$$\therefore\ y^/ = \frac{(-4y^3 + 2xy - 3x^2 + 5)}{(12xy^2 - x^2)} \quad . \ \S\S\S$$

The slope: $m = y^/\Big]_{(1,1)} = \left.\frac{(-4y^3 + 2xy - 3x^2 + 5)}{(12xy^2 - x^2)}\right]_{(1,-1)} = \frac{4}{11} \quad . \ \S\S\S$

(b) $y = x^2 \sin y \quad \Rightarrow \quad y^/ = 2x.\sin y + x^2 \cos y.y^/$

$$\Rightarrow 1 - x^2 \cos y\ y^/ = 2x.\sin y$$

$$\therefore\ y^/ = \frac{2x.\sin y}{1 - x^2 \cos y} \quad . \ \S\S\S$$

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Example : 18

Find $y^/$ for the function $xy = \tan y$

Solution:

$$xy = \tan y \quad \Rightarrow \quad x.y^/ + 1.y = \sec^2 y.y^/$$

$$(x - \sec^2 y)y^/ = -y \quad \Rightarrow \quad y^/ = -y/(x - \sec^2 y) \quad . \ \S\S\S$$

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Example : 19

Find $y^/$ for the function $x^2 + \sqrt{\sin y} - y^2 = 1$

Solution:

$$2x + \frac{1}{2\sqrt{\sin y}}\, y^/ - 2yy^/ = 0 \quad \Rightarrow \quad x^2 + \sqrt{\sin y} - y^2 = 1$$

$$\Rightarrow y^/ = -2x \Big/ \left(\frac{1}{2\sqrt{\sin y}} - 2y\right) \quad \Rightarrow \quad \left(\frac{1}{2\sqrt{\sin y}} - 2y\right).y^/ = -2x \quad . \ \S\S\S$$

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E x e r c i s e ***(4-2)***

(I) Find $y^{/}$ for the following functions **:**

(1) $y=(7x+\sqrt{x^2+3})^6$, (2) $y=\sqrt[3]{8x^3+27}$

(3) $y=(2x^2-9x+8)^{-2/3}$, (4) $y=\sqrt[3]{8x^3+27}$

(5) $y=\sec^3 2x$, (6) $y=\tan^3 4x$

(7) $y=\sec(2x+1)^2$, (8) $y=\sqrt{\sin 6x}.\sqrt{\cos x}$

(9) $y=(\sin 3x+\cos 3x)^2$, (10) $y=\cot(x^3-2x)$

(11) $y=\dfrac{\sec^3 x}{1+\csc^2 x}$, (12) $y=\dfrac{2x+3}{\sqrt[5]{4x^2+9}}$

(13) $y=\dfrac{\cos 4x}{1-\sin 4x}$, (14) $y=\tan\sqrt[3]{5-6x}$

(15) $y=(\sin 3x+\cos 3x)^2$, (16) $y=\cot(x^3-2x)$

(17) $y=\sin\sqrt{x}+\sqrt{\sin x}$, (18) $y=\sec[\text{ sex } 2x]$

(II) Find $y^{/}$ for the following functions using the chain rule

(1) $x=\dfrac{t+1}{t-1}$; $y=t^2+1$, (2) $x=2t$; $y=t^2+3$

(3) $y=\tan 3u$; $u=x^2$, (4) $y=1/u$; $u=\sqrt{3x-2}$

(5) $x=a\cos 2t^2$; $y=b\sin t$, (6) $y=u\sin u$; $u=x^3$

(III) Find $y^{/}$ for the following implicit functions

(1) $5x^2+2x^2y+y^2=8$, (2) $x=\sin(xy)$

(3) $\sin^2 3y=x+y-1$, (4) $x^4+4x^2y^2-3xy^3+2x=0$

(5) $\sqrt{x}+\sqrt{y}=100$, (6) $y^2+1=x^2\sec y$

(IV) Find $y^{//}$ for the following functions(if exist)

(1) $\sin y+y=x$, (2) $x^2y^3=1$

(3) $x^3-y^3=1$, (4) $5x^2-2y^2=4$

(5) $\cos y=x$, (6) $xy^2+3y=27$

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I-7 Derivatives Of A Logarithmic Functions

In this section we will discuss the relation between the general $\log_a x$ and natural $\ln x$ logarithmic function and the rules of derivatives for them.

The general logarithmic function denoted by $\log_a x$,where $a \in R^+$ and called the general base of the logarithm, if a is equal to the natural base and called $\log_e x = \ln x$ we get , $e \approx 2.71828$ natural logarithmic function .

There exist different between the numerical value of them shown in the following algebraic relation:

$$\log_a x = \frac{1}{\ln a} \ln x$$

First, we will introduce the rules of differentiation for the **natural exponential function** and using the previous relation the general exponential function can be differentiate.

RULE **: 7**

$$D_x \ln x = \frac{1}{x}$$

Theorem **: (4-4)**

If $g(x)$ is differentiable then:

(1) $D_x \ln g(x) = \frac{1}{g(x)} . g^{/}(x) \quad ; \quad g(x) > 0$

(2) $D_x \ln|g(x)| = \frac{1}{g(x)} . g^{/}(x) \quad ; \quad g(x) \neq 0$

Example **: 20**

If $f(x)=\ln(x^2-6)$ find $f^{/}(x)$

Solution:

$$f^{/}(x)=D_x\ln\underbrace{(x^2-6)}_{g(x)}=\frac{1}{(x^2-6)}.\underbrace{D_x(x^2-6)}_{g^{/}(x)}$$

$$\therefore\quad f^{/}(x)=\frac{1}{(x^2-6)}(2x)=\frac{2x}{(x^2-6)}\quad .$$ §§§

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Example **: 21**

If $y=\ln\sqrt{x+1}$ find $y^{/}$.

Solution:

$$y^{/}=\frac{d}{dx}\ln\underbrace{\sqrt{x+1}}_{g(x)}=\frac{1}{\sqrt{x+1}}\frac{d}{dx}\sqrt{x+1}$$

$$=\frac{1}{\sqrt{x+1}}.\frac{1}{2}(x+1)^{-1/2}$$

$$\therefore\ y^{/}=\frac{1}{\sqrt{x+1}}.\frac{1}{2}\frac{1}{\sqrt{x+1}}=\frac{1}{2(x+1)}\quad .$$ §§§

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Example **: 22**

If $f(x)=\ln(4+5x-2x^3)$ find $f^{/}(x)$.

Solution:

$$f^{/}(x)=D_x\ln\underbrace{(4+5x-2x^3)}_{g(x)}$$

$$=\frac{1}{(4+5x-2x^3)}D_x(4+5x-2x^3)$$

$$=\frac{1}{(4+5x-2x^3)}.(5-6x^2)$$

$$\therefore\quad f^{/}(x)=\frac{(5-6x^2)}{(4+5x-2x^3)}\quad .$$ §§§

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Rules For Natural Logarithmic Function

If $p>0$, $q>0$ then :

(1) $\ln pq = \ln p + \ln q$, (2) $\ln \frac{p}{q} = \ln p - \ln q$, (3) $\ln p^r = r \ln p$

e.g.

$f(x)$	$f(x)$ **Using log rule**	$f'(x)$
► $\ln[(x+2)(3x-5)]$	$\ln(x+2)+\ln(3x-5)$	$\frac{1}{(x+2)}+\frac{3}{(3x-5)}$
► $\ln\frac{(x+2)}{(3x-5)}$	$\ln(x+2)-\ln(3x-5)$	$\frac{1}{(x+2)}-\frac{3}{(3x-5)}$
► $\ln(x^2+2)^5$	$5\ln(x^2+2)$	$5.\frac{1}{(x^2+2)}.2x$
► $\ln\sqrt{(x+2)}$	$\frac{1}{2}\ln(x+2)$	$\frac{1}{2}.\frac{1}{(x+2)}$

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Example **: 23**

If $f(x)=\ln\left[\sqrt{6x-1}(4x+5)^3\right]$ find $f'(x)$.

Solution:

$$f(x)=\ln\left[(6x-1)^{1/2}.(4x+5)^3\right]=\frac{1}{2}\ln(6x-1)+3\ln(4x+5)$$

$$f'(x)=\frac{1}{2}.\frac{1}{(6x-1)}.6+3.\frac{1}{(4x+5)}.4$$

$$=\frac{3}{(6x-1)}+\frac{12}{(4x+5)} .$$ §§§

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Example **: 24**

If $y=\ln\sqrt[3]{\frac{x^2-1}{x^2+1}}$ find $\frac{dy}{dx}$.

Solution:

$$y=\ln\left(\frac{x^2-1}{x^2+1}\right)^{1/3}=\frac{1}{3}\ln\left(\frac{x^2-1}{x^2+1}\right)=\frac{1}{3}\left(\ln(x^2-1)-\ln(x^2+1)\right)$$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{3}\left(\frac{1}{(x^2-1)}.2x-\frac{1}{(x^2+1)}.2x\right)=\frac{2x}{3}\left(\frac{1}{(x^2-1)}.-\frac{1}{(x^2+1)}\right)=\frac{4x}{3(x^4-1)} .$$ §§§

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***Example* : 25**

If $y=\frac{(5x-4)^3}{\sqrt{2x+1}}$ find $D_x y$.

Solution:

Take the natural logarithm for both sides:

$$\ln y=\ln\frac{(5x-4)^3}{\sqrt{2x+1}}=3\ln(5x-4)-\frac{1}{2}\ln(2x+1)$$

Differentiate the both sides:

$$\frac{1}{y}y'=3\frac{1}{(5x-4)}.5-\frac{1}{2}.\frac{1}{(2x+1)}.2$$

$$\Rightarrow \quad y'=y\left[\frac{15}{(5x-4)}-.\frac{1}{(2x+1)}\right]$$

$$\therefore \quad D_x y=y'=\left(\frac{(5x-4)^3}{\sqrt{2x+1}}\right).\left[\frac{15}{(5x-4)}-\frac{1}{(2x+1)}\right] \quad .$$ §§§

***Theorem* : (4-5)**

(1) $D_x \log_a x=D_x\left[\frac{\ln x}{\ln a}\right]=\frac{1}{\ln a}D_x[\ln x]$

(2) $D_x \log_a f(x)=D_x\left[\frac{\ln f(x)}{\ln a}\right]=\frac{1}{\ln a}D_x[\ln f(x)]$

***Example* : 26**

If $f(x)=\log\sqrt[3]{(2x+5)^2}$ find $f'(x)$

Solution:

$$f(x)=\log\sqrt[3]{(2x+5)^2}=\frac{1}{\ln 10}\left[\ln\sqrt[3]{(2x+5)^2}\right]$$

$$=\frac{1}{\ln 10}\left[\ln(2x+5)^{2/3}\right]=\frac{1}{\ln 10}\left[\frac{2}{3}\ln(2x+5)\right]$$

$$\therefore \quad f'(x)=D_x f(x)=\frac{1}{\ln 10}.D_x\left[\frac{2}{3}\ln(2x+5)\right]$$

$$=\frac{1}{\ln 10}.\left[\frac{2}{3}.\frac{2}{(2x+5)}\right]=\frac{1}{\ln 10}.\left[.\frac{4}{3(2x+5)}\right].$$ §§§

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Example **: 27**

If $f(x)=\log_5\left|\frac{6x+4}{2x-3}\right|$ find $f^{/}(x)$

Solution:

$$f(x)=\log_5\left|\frac{6x+4}{2x-3}\right|=\frac{1}{\ln 5}\left(\ln[6x+4]-\ln[2x-3]\right)$$

$$\therefore \quad f^{/}(x)=\frac{1}{\ln 5}\left(\frac{6}{(6x+4)}-\frac{2}{(2x-3)}\right) \ .$$ §§§

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I-8 **Derivatives Of Exponential Functions**

Theorem **: (4-6)**

(1) $\ln e^{f(x)}=f(x)$; $\forall \ x\in R$

(2) $e^{\ln f(x)}=f(x)$; $\forall \ x\in R$

(3) $\ln e=1$

e.g.

► $\ln e^{5x}=5x$, ► $\ln e^{\sqrt{2x+1}}=\sqrt{2x+1}$, ► $e^{3\ln x}=e^{\ln x^3}=x^3$.

Rules Of Natural Power

If p , q are real numbers and r is rational number :

(i) $e^p e^q=e^{p+q}$, (ii) $\frac{e^p}{e^q}=e^{p-q}$, (iii) $(e^p)^r=e^{pr}$

First we will introduce the rules of differentiation of the natural exponential function (e^x).

Theorem **: (4-7)**

$D_x e^x=e^x$

Theorem **: (4-8)**

If $f(x)$ is differentiable then: $D_x(e^{f(x)})=e^{f(x)}.f^{/}(x)$

***Example* : 28**

If $y = e^{\sqrt{x^2+1}}$ find. y'

Solution:

$$y' = \frac{dy}{dx} = \frac{d}{dx} e^{\overbrace{\sqrt{x^2+1}}^{f(x)}} = e^{\sqrt{x^2+1}} \overbrace{\frac{d}{dx}(x^2+1)^{1/2}}^{f'(x)}$$

$$= e^{\sqrt{x^2+1}} \cdot \overbrace{\frac{1}{2}(x^2+1)^{-1/2}.2x}^{f'(x)}$$

$$= e^{\sqrt{x^2+1}} \cdot \frac{x}{\sqrt{x^2+1}} = \frac{x\, e^{\sqrt{x^2+1}}}{\sqrt{x^2+1}} \quad .$$ §§§

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***Example* : 29**

If $y = e^{\sqrt{x}} + \sqrt{e^x}$ find. y'

Solution:

$$y = e^{\sqrt{x}} + \sqrt{e^x} = e^{x^{1/2}} + (e^x)^{1/2}$$

$$y' = e^{x^{1/2}} . D_x(x^{1/2}) + \frac{1}{2}(e^x)^{-1/2} . D_x(e^x)$$

$$= e^{x^{1/2}} . \frac{1}{2} x^{-1/2} + \frac{1}{2}(e^x)^{-1/2}.(e^x.1)$$

$$= \frac{e^{\sqrt{x}}}{2\sqrt{x}} + \frac{1}{2}\sqrt{e^x} \quad .$$ §§§

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***Example* : 30**

If $y = e^{\ln\sin 5x}$ find. y'

Solution:

$$y = e^{\ln\sin 5x} \Rightarrow y' = 5\cos 5x \quad \therefore \quad \Rightarrow \quad y = \sin 5x \quad .$$ §§§

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***Example* : 31**

If $y = \ln(\csc e^{3x})$ find. y'

Solution:

$$y = \ln(\csc e^{3x}) \quad \Rightarrow \quad y' = \frac{1}{(\csc e^{3x})}.D_x(\csc e^{3x})$$

$$\therefore \quad y' = \frac{1}{(\csc e^{3x})}.(-\csc e^{3x}.\cot e^{3x}.e^{3x}.3) = -3e^{3x}\cot e^{3x} \quad .$$ §§§

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General Exponential Function (a^x)

$$a^x = e^{x\ln a} \quad ; \quad \forall\ a > 0 \ , \quad x \in R$$

Rules Of General Exponential Function

If $a > 0$, $b > 0$ and u , v real numbers then:

(1) $a^u a^v = a^{u+v}$, (2) $(a^u)^v = a^{u.v}$, (3) $(ab)^u = a^u b^u$

(4) $\dfrac{a^u}{a^v} = a^{u-v}$, (5) $\left(\dfrac{a}{b}\right)^u = \dfrac{a^u}{b^u}$

Theorem **: (4-9)**

(1) $D_x a^x = a^x \ln a$

(2) $D_x a^{f(x)} = a^{f(x)}.f^{/}(x).\ln a$

Example **: 32**

If $y = (x^2+1)^{10} + 10^{x^2+1}$ find $y^{/}$

Solution:

$$y = (x^2+1)^{10} + 10^{x^2+1}$$

$$y^{/} = \underbrace{10.(x^2+1)^9.(2x)} + \underbrace{10^{x^2+1}.(2x).\ln 10}\ .$$

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Differentiate Function Of The Form $y=\{f(x)\}^{g(x)}$

We will apply the following steps:

1- Take natural logarithm for both sides of the equation:

$$\ln y=\ln\{f(x)\}^{g(x)} \Rightarrow \ln y=g(x).\ln\{f(x)\}$$

2- Differentiate the both sides:

$$D_x \ln y=D_x[g(x).\ln\{f(x)\}]$$

$$\Rightarrow \quad \frac{1}{y}.y^{/}=\left[g^{/}(x).\ln\{f(x)\}+\frac{1}{f(x)}.f^{/}(x).g(x)\right]$$

3- Multiply the both sides of the equation by y and replace it from the question:

$$y^{/}=y\left[g^{/}(x).\ln\{f(x)\}+\frac{1}{f(x)}.f^{/}(x).g(x)\right]$$

$$y^{/}=\{f(x)\}^{g(x)}.\left[g^{/}(x).\ln\{f(x)\}+\frac{1}{f(x)}.f^{/}(x).g(x)\right]$$

Example **: 33**

If $y=(\cos 2x)^{x}$ find. $y^{/}$

Solution:

Take ($\ln$) for both sides

$$\ln y=\ln\left[(\cos 2x)^{x}\right] \Rightarrow \ln y=\underbrace{x}_{u}.\underbrace{\ln\cos 2x}_{v}$$

$$\Rightarrow \quad \frac{1}{y}.y^{/}=\underbrace{x}_{u}.\underbrace{\frac{1}{\cos 2x}(-2\sin 2x)}_{v^{/}}+\underbrace{1}_{u^{/}}.\underbrace{\ln\cos 2x}_{v}$$

$$\Rightarrow \quad y^{/}=y.[-2x\tan 2x+\ln\cos 2x]$$

$$\Rightarrow \quad y^{/}=(\cos 2x)^{x}.[-2x\tan 2x+\ln\cos 2x] \text{ . §§§}$$

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Exercise *(4-3)*

(I) Find $f^{/}(x)$ for the following functions:

(1) $f(x)=\ln\sqrt{x}+\sqrt{\ln x}$, (2) $f(x)=\dfrac{1}{\ln x}+\ln\dfrac{1}{x}$

(3) $f(x)=\ln\dfrac{\sqrt{x^2+1}}{(9x-4)^2}$, (4) $f(x)=\ln\sqrt{\dfrac{4+x^2}{4-x^2}}$

(5) $f(x)=\log_5\left|\dfrac{1-x^2}{2-5x^3}\right|$, (6) $f(x)=\log_3\ln x$

(II) Find $f^{/}(x)$ for the following functions:

(1) $f(x)=\ln\cos e^{-x}$, (2) $f(x)=8^{-3x}\cos 3x$

(3) $f(x)=\ln e^{\tan(3x+1)}$, (4) $f(x)=e^{1/x}+(1/e^{x})$

(5) $f(x)=\sec^2(e^{-4x})$, (6) $f(x)=e^{x\ln x}$

(7) $f(x)=\sqrt{e^{3x}+e^{-3x}}$, (8) $f(x)=5^{3x}+(3x)^5$

(9) $f(x)=\csc^5 3^{-2x}$, (10) $f(x)=1/(\sin^2 8^{-2x})$

(III) Find $y^{/}$ for the following functions:

(1) $y=(\sin x)^{\cos x}$, (2) $y=(10^{x}+10^{-x})^{10}$

(3) $y=x^{e}+e^{x}$, (4) $y=2^{\sin^2 x}$

(5) $y=(x^2+1)^{2x}$, (6) $y=x^{\tan x}$

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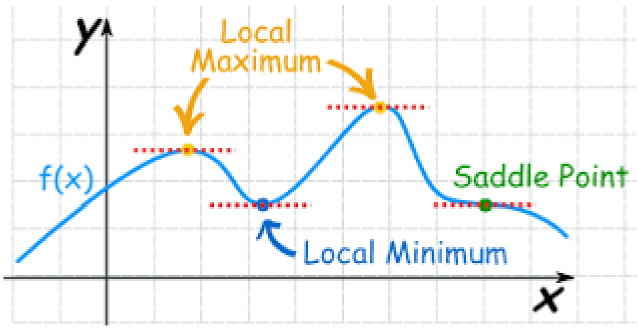
II- DERIVATIVES APPLICATIONS

II-1 Maximum And Minimum Values Of A Functions

In many applications, it is useful to determine where a function takes its maximum and minimum values.

A function *f* has local maximum (or relative maximum) at $x = x_1$ if there is an interval around x_1 on which $f(x_1) \geq f(x)$ for all x in that interval, see Figure . A function f has local minimum (or relative minimum) at $x = x_2$ if there is an interval around x_2 on which $f(x_2) \leq f(x)$ for all x in that interval, see Figure



Theorem **: (4-9) (The Extreme Value Theorem)**

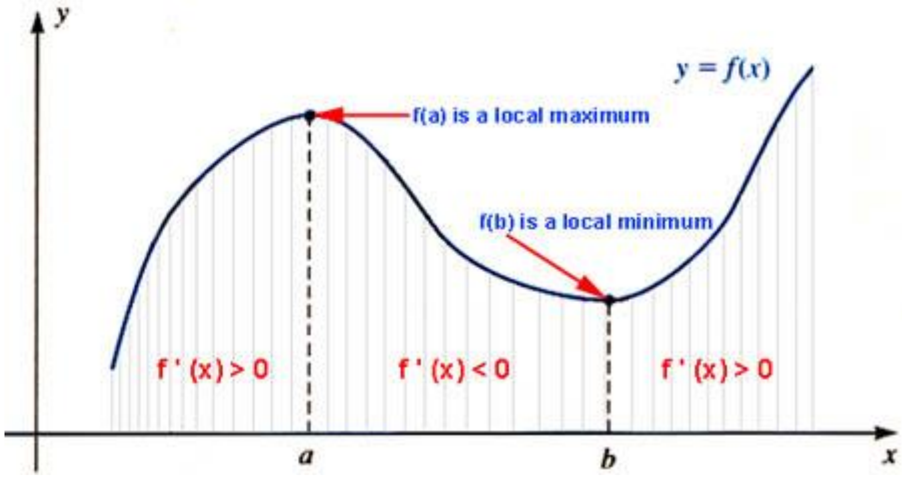
If a function f is continuous on closed interval [*a*, *b*], then f contains its absolute maximum and absolute minimum at some points in [a, b].

Definition **:** *(4.2)*)

A critical number of a function f is a number *c* in the domain of f such that either $f^{/}(c) = 0$ or $f^{/}(c)$ does not exist.

Theorem **: (4-10)**

If f has local maximum or minimum at number $x = c$, then $f^{/}(c) = 0$ or $f^{/}(c)$ does not exist.



Show local extrema of a function

The Closed Interval Method

Guidelines for finding the extrema of continuous function $f(x) on\ [a, b]$

1. Find all critical of $f(x) on\ (a, b)$, i.e. the points at which either $f^{/}(c) = 0$ or $f^{/}(c)$ does not exist.
2. Calculate $f(c)$for each critical number c found in step 1.
3. Calculate the endpoints values $f(a) and\ f(b)$.
4. The absolute maximum and minimum values calculated in step 2 &3.

Example **: 34**

Find the absolute maximum and absolute minimum of the function

$f(x) = 2x^3 + 3x^2 - 12x - 7$ on $-3 \leq x \leq 0$

Solution:

Since $f(x)$ is continuous on R**.** We can use **closed interval method**

$f(x) = 2x^3 + 3x^2 - 12x - 7 \rightarrow f^{/}(x) = 6x^2 + 6x - 12$

Since $f^{/}(x)$ exist for all x, the only critical numbers of f occur when

$f^{/}(x) = 0 \rightarrow (x + 2)(x - 1) = 0 \rightarrow x = -2\ and\ x = 1$. Of these, only $x = -2$ lies in the interval $-3 \leq x \leq 0$. Compute $f(x)$ at $x = -2$ and at the endpoints $x = -3$ and $x = 0$

$f(-2) = 13$ $f(-3) = 2$ $f(0) = -7$

Then the absolute maximum of $f(x)$ on interval $-3 \leq x \leq 0$ is $f(-2) = 13$, and the absolute minimum is $f(0) = -7$.§§§

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II-2 Increasing And Decreasing Functions

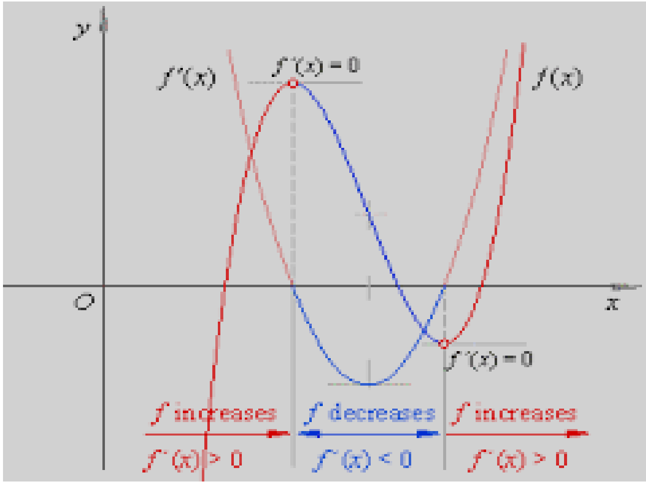
Definition **:** *(4.3)*

The Increasing and Decreasing Test (I/D test)

Let $f(x)$ be continuous and differentiable on $[a,b]$

$i.\ f^{/}(x) > 0\ for\ every\ x\ in\ (a,b),\ then\ f(x) is\ increasing\ on\ [a,b]$

$ii. f^{/}(x) < 0\ for\ every\ x\ in\ (a,b),\ then\ f(x) is\ decreasing\ on\ [a,b]$



Increasing and decreasing function

Example **: 35**

Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where is decreasing**.**

Solution:

Applying the (I/D test), we have

$$f^{/}(x) = 12x^3 - 12x^2 - 24x \quad = 0$$

$$\Rightarrow \qquad = 12x(x^2 - x \quad - 2) = 0 \rightarrow x(x-2)(x+1) = 0$$

Then $f^{/}(x) = 0$ at $x = 0, x = -1\ \&x = 2$

The following table shows exactly the increasing and decreasing intervals of the given function

intervals	$(-\infty,-1)$	$[-1,0)$	$[0,2)$	$[2,\infty)$
Sing $f^{/}(x)$	-ve	+ ve	-ve	+ve
$f(x)$	decreasing	Increasing	decreasing	increasing

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Definition : (4.43) (The First Derivative Test)

Let $f(x)$ be continuous and C is critical number of (x) .

** If $f^{/}$ changes from positive to negative, then f has a local maximum at C

** If $f^{/}$ changes from negative to positive, then f has a local minimum at C

** If $f^{/}$ does not change sign at C, then f has no local extrema at C

Example : 36

Finding Local Extrema of the function

$$f(x) = 2x^3 + 9x^2 - 24x - 10$$

Solution:

Applying the (I/D test), we have

$$f^{/}(x) = 6x^2 + 18x - 24 = 0$$

$$= 6(x^2 + 3x - 4) = 0 \rightarrow (x+4)(x-1) = 0$$

Then $f^{/}(x) = 0$ at $x = -4$ & $x = 1$

The following table shows exactly the increasing, decreasing intervals of the given function and local extrema points.

intervals	$(-\infty, -4)$	$[-4,1)$	$[1,\infty)$
Sing $f^{/}(x)$	+ve	-ve	+ve
$f(x)$	increasing	decreasing	increasing

Then $x = -4$ is local maximum and $x = 1$ is local minimum. §§§

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Curve Sketching

Some particular aspects of curve sketching, domains, range, symmetry, limits, continuity, extreme values, I/D test, and first derivative test. In this section, we put all this information together to sketch graphs that reveal the important features of the function.

Example **: 37**

If $f(x) = x^{\frac{1}{3}}(8-x)$ find its local extrema of $f(x)$ and sketch its graph

Solution:

Step1: find all Critical numbers of $f(x)$

We have $f^{/}(x) = -x^{\frac{1}{3}} + \frac{1}{3}x^{\frac{-2}{3}}(8-x) = \frac{-3x+8-x}{3x^{\frac{2}{3}}} = \frac{-4x+8}{3x^{\frac{2}{3}}}$

So $f^{/}(x) = 0$ if $x = 2$ & $f^{/}(x)$ does not exist when $x = 0$

Thus the critical numbers are $x = 0$ &2 §§§

Step 2: Apply the first derivative test that is

intervals	$(-\infty, 0)$	$[0,2)$	$[2,\infty)$
Sing $f^{/}(x)$	+ve	+ve	-ve
$f(x)$	increasing	increasing	decreasing

the first derivative test implies that $f(x)$ has local maximum at $x = 2$, and does not has any extrema at $x = 0$ §§§

Step3: Sketch the graph

i. Find the $x -$ intercepts of the function, i.e. the points at which $y = 0$

$x^{\frac{1}{3}}(8-x) = 0 \rightarrow x = 0$ & $x = 8$

ii. Plot the points corresponding to critical numbers $x = 0$ &2

That the graph has a vertical tangent at $(0,0)$ §§§

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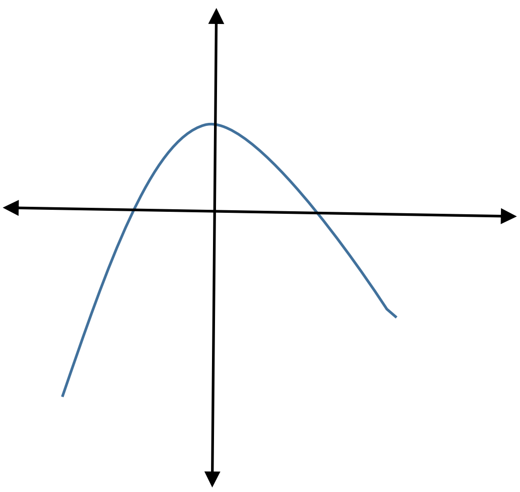

Exercise (4-4)

I. Find the absolute maximum and absolute minimum if they exist of the following functions

1. $f(x) = x^3 - 3x^2 + 1, -\frac{1}{2} \leq x \leq 4$
2. $f(x) = x^2 + \frac{16}{x}, 1 \leq x \leq 5$
3. $f(x) = 3x^4 - 16x^3 + 18x^2, -1 \leq x \leq 4$

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CHAPTER 5

INVERSE TRIGONOMETRIC AND INVERSE HYPERBOLIC FUNCTIONS

In this chapter, we will retrieve the ***definition of inverse trigonometric functions***, which have already been studied in the trigonometry course, and then ***introduce the derivative and integral laws*** of these functions, which help in the study of some physical applications such as determining the angle of fall of light on the surface and study the rate of change in the angle of fall and Effect on light intensity.

Also , a definition ***hyperbolic functions***, which has a behavior very similar to trigonometric functions , ***BUT*** ,different from them as they can not be represented on a triangle but study for their important applications in physical and engineering applications such as determining the velocity of an object in a resistive medium such as water or air

I INVERSE TRIGONOMETRIC FUNCTIONS

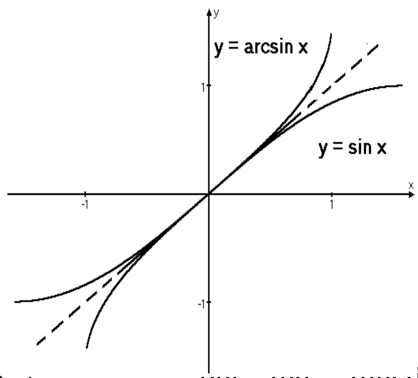
I-1 Inverse Trigonometric Functions Overview

Trigonometric functions are not one to one functions, so inverse functions can not be found, but by putting some limitations on the domain of these functions we can then define the inverse of these functions, and also study the laws of differentiation and integration of the inverse trigonometric functions.

Definition **: (5.1) (** inverse sine **)**

The inverse sine function is denoted by $\sin^{-1} x$ or , ($\arcsin x$) and it is defined by :

$$y = \sin^{-1} x \quad \Leftrightarrow \quad x = \sin y \ \forall -1 \le x \le 1 \ , \ -\pi/2 \le y \le \pi/2$$



N O T E

** the inverse sine function $\sin^{-1} x \ne \dfrac{1}{\sin x}$, ***while*** $(\sin x)^{-1} = \dfrac{1}{\sin x}$, but the function $\sin^{-2} x$ can be written $(\sin x)^{-2} = \dfrac{1}{\sin^2 x} = \dfrac{1}{(\sin x)^2}$.

Similarly , all inverse trigonometric functions where the exponent for is reserved only ($^{-1}$) the use of inverse trigonometric function.

** Inverse sine function $\sin^{-1}$, also symbolized , meaning that $\arc\sin$ we can use $\arc\sin x$ instead of. $\sin^{-1} x$

**)1/2) is the angle that has sin (y phrase reads that his t ($y = \sin^{-1}(1/2)$ that is the angle $(\pi/6)$)

The Properties of ($\sin^{-1}$)

(i) $\sin(\sin^{-1} x) = x \;;\; -1 \le x \le 1$

(ii) $\sin^{-1}(\sin x) = x \;;\; -\pi/2 \le x \le \pi/2$

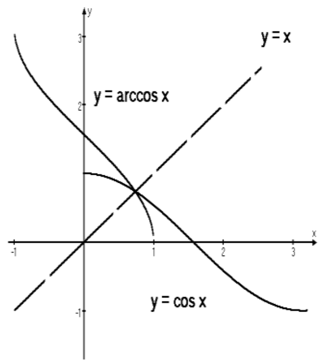
All inverse trigonometric functions has the same futures and same properties like $\sin^{-1}$ which can be summarized as follows :

Definition **: (5.2) (inverse cosine)**

The inverse cosine function denoted by $\cos^{-1}$(arcos) and it is defined by :

$y = \cos^{-1} x \iff x = \cos y$

, $\forall -1 \le x \le 1$ and $0 \le y \le \pi$



The Properties of ($\cos^{-1}$)

(i) $\cos(\cos^{-1} x) = x \;;\; -1 \le x \le 1$

(ii) $\cos^{-1}(\cos x) = x \;;\; 0 \le x \le \pi$

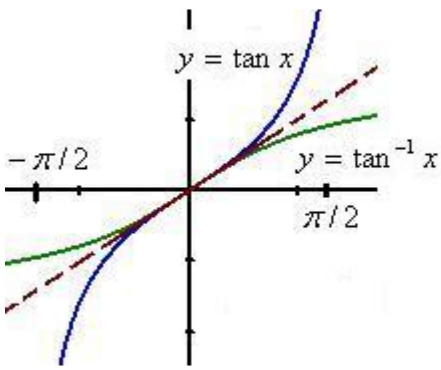
Definition **: (5.3) (inverse tan)**

The inverse tangent function denoted by $\tan^{-1}$,

And it is defined by :

$y = \tan^{-1} x \iff x = \tan y$,

$\forall x \in R$ and $-\pi/2 \le y \le \pi/2$



The Properties of ($\tan^{-1}$)

(i) $\tan(\tan^{-1} x) = x \;;\; \forall\, x \in R$

(ii) $\tan^{-1}(\tan x) = x \;;\; -\pi/2 \le x \le \pi/2$

I-2 Derivatives Rules Of Inverse Trigonometric Functions

In this section we will focus on the study of the ***differentiation and integration of inverse triangular functions*** ($sin^{-1}, cos^{-1}, tan^{-1}\ sec^{-1}, \ldots$). This derivative rules are given in the following list:

Derivative Rules For Inverse Trigonometric Functions)

If $u = g(x)$ is a differentiable function , then

(1) $D_x \sin^{-1} u = \dfrac{1}{\sqrt{1-u^2}} D_x u$

(2) $D_x \cos^{-1} u = -\dfrac{1}{\sqrt{1-u^2}} D_x u$

(3) $D_x \tan^{-1} u = \dfrac{1}{1+u^2} D_x u$

(4) $D_x \sec^{-1} u = \dfrac{1}{u\sqrt{u^2-1}} D_x u$

Illustrative examples

$f(x)$	$f'(x)$
* $\sin^{-1} \underbrace{3x}_{u}$	$\dfrac{1}{\sqrt{1-(3x)^2}} D_x(3x) = \dfrac{3}{\sqrt{1-(3x)^2}}$
* $\cos^{-1} \underbrace{\ln x}_{u}$	$-\dfrac{1}{\sqrt{1-(\ln x)^2}} D_x(\ln x) = -\dfrac{3}{x\sqrt{1-(\ln x)^2}}$
* $tan^{-1} \underbrace{e^{2x}}_{u}$	$\dfrac{1}{1+(e^{2x})^2} D_x(e^{2x}) = \dfrac{2e^{2x}}{1+e^{4x}}$
* $\sec^{-1} \underbrace{x^2}_{u}$	$\dfrac{1}{x^2\sqrt{(x^2)^2-1}} D_x(x^2) = \dfrac{2}{x\sqrt{x^4-1}}$

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***Example* : 1**

Find the first derivative $y^{/}$ for each of the following functions:

(a) $y = \tan^{-1}\sin 2x$, (b) $y = \cos^{-1}\cos e^{3x}$.

(c) $y = \cos(x^{-1}) + (\cos x)^{-1} + \cos^{-1}x$, (d) $y = (\tan x)^{\tan^{-1}x}$.

Solution:

(a) If : $y = \tan^{-1}\sin 2x \Rightarrow y = \tan^{-1}\underbrace{\sin 2x}_{u}$

$$y^{/} = \frac{1}{1+(\sin 2x)^2}.D_x(\sin 2x)$$

$$\Rightarrow \quad y^{/} = \frac{1}{1+(\sin 2x)^2}.2\cos 2x = \frac{2\cos 2x}{1+(\sin 2x)^2} \quad .§§§$$

(b) If : $y = \cos^{-1}\cos e^{3x} \Rightarrow y = e^{3x}$

$$y^{/} = 3e^{3x} \quad .§§§$$

(c) If : $y = \cos(x^{-1}) + (\cos x)^{-1} + \cos^{-1}x$

$$y^{/} = \left[-\sin(x^{-1}).(-x^{-2})\right] + \left[-(\cos x)^{-2}.(-\sin x)\right] + \frac{1}{\sqrt{1-\cos^2 x}}$$

$$= \left[\frac{1}{x^2}\sin(x^{-1})\right] + \left[\frac{\sin x}{\cos^2 x}\right] - \frac{1}{\sqrt{1-\cos^2 x}}$$

$$= \frac{\sin(x^{-1})}{x^2} + \tan x.\sec x - \frac{1}{\sqrt{1-\cos^2 x}} \quad .§§§$$

(d) If : $y = (\tan x)^{\tan^{-1}x}$

By taking a logarithm for both sides of the equation we find that

$$\ln y = \underbrace{(\tan^{-1}x)}_{u}.\underbrace{(\ln \tan x)}_{v}$$ (***Hint:*** $D_x(u.v) = u.v^{/} + u^{/}.v.$)

Differentiate the two sides of the equation as a multiplication of two functions

$$\frac{1}{y}y^{/} = \underbrace{\tan^{-1}x}_{u}.\underbrace{\left[\frac{1}{\tan x}.\sec^2 x\right]}_{v^{/}} + \underbrace{\left[\frac{1}{1+x^2}\right]}_{u^{/}}.\underbrace{\ln(\tan x)}_{v}$$

$$y^{/} = \underbrace{(\tan x)^{\tan^{-1}x}}_{y}.\left[\tan^{-1}x.\frac{\sec^2 x}{\tan x} + \frac{\ln(\tan x)}{1+x^2}\right] \quad .§§§$$

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Exercise (5-1)

Inverse Trigonometric Functions

I- Find $y^{/}$ for all the following functions :

(1) $y = x^2 \operatorname{arcsec} x^2$, (2) $y = \sqrt{\sin^{-1}(1-x^2)}$

(3) $y = \sin^{-1}\sqrt{(1-x^2)}$, (4) $y = (\tan x + \tan^{-1} x)^2$

(5) $y = \tan^{-1}\sqrt{\tan 2x}$, (6) $y = \tan^{-1}(\tan^{-1} x)$

(7) $y = 10^x \log x$, (8) $y = \dfrac{\tan^{-1} x}{1+x^2}$

(9) $y = 3^{\arcsin(x^3)}$, (10) $y = e^{4x} \operatorname{arcsec} e^{4x}$

(11) $y = \sec^{-1}\sqrt{x^2-1}$, (12) $y = (1+\cos^{-1} 3x)^2$

(13) $y = x^2 \sec^{-1} 5x$, (14) $y = (\sin 2x)(\sin^{-1} 2x)$

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II. Hyperbolic Functions

The two expressions $\frac{e^x - e^{-x}}{2}$, $\frac{e^x + e^{-x}}{2}$ used in the sense of differentiation and integration have the same characteristics of the two trigonometric functions $\sin x$, $\cos x$ respectively , and were named ***hyperbolic sine function*** and ***hyperbolic cosine*** function for x.

Definition **: (5.4)**

Hyperbolic sine function of variable x is denoted by and the) shx (or for simplicity $\sinh x$ hyperbolic cosine function of variable x is denoted byand) chx (or for simplicity $\cosh x$ defined by :

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad , \quad \cosh x = \frac{e^x + e^{-x}}{2} \quad , \quad \forall x \in R$$

In similar , way we can represent the other hyperbolic function of behavior between trigonometric and hyperbolic functions has helped us to define the remaining ratios of hyperbolic functions as follows

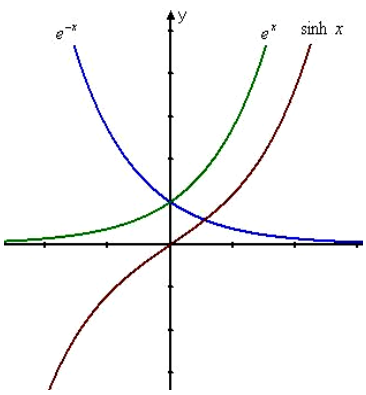
II-1 Hyperbolic Functions Representation

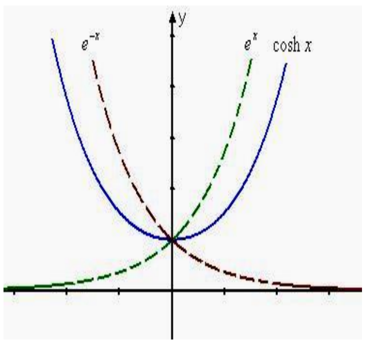
(1) $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

(2) $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$, $x \neq 0$

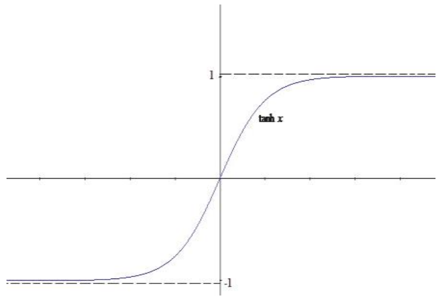
(3) $\text{sech } x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

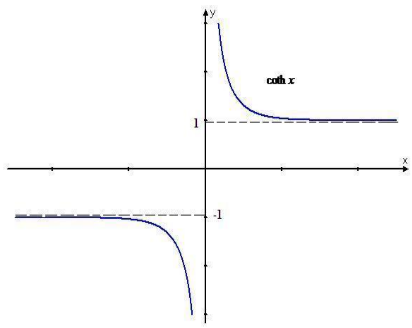
(4) $\text{csch } x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$, $x \neq 0$



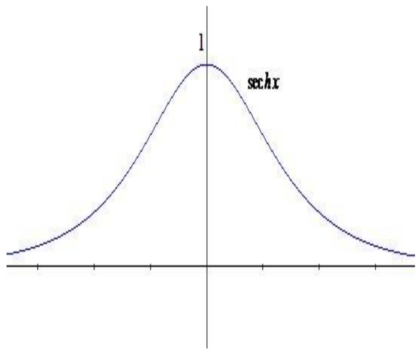


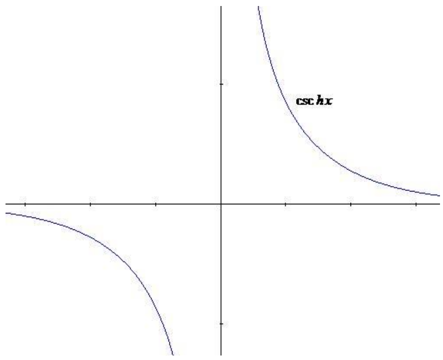
$y = \sinh x$ $\qquad\qquad$ $y = \cosh x$





$y = \tanh x$ $\qquad\qquad$ $y = \coth x$





$y = \text{sec}h\,x$ $\qquad\qquad$ $y = \text{csc}h\,x$

By using the definitions of hyperbolic functions , we can easily define the conclusion of hyperbolic function relationships in the following theory:

Hyperbolic Functions Identities

(i) $1-\tanh^2 x = \text{sec}\,h^2 x$, (ii) $\cosh^2 x - \sinh^2 x = 1$, (iii) $\coth^2 x - 1 = \text{csc}\,h^2 x$

II-2 Derivative Rules For Hyperbolic Functions

If $u = g(x)$ is a differentiable function , then

(1) $D_x \sinh u = \cosh u \,.\, D_x u$

(2) $D_x \cosh u = \sinh u \,.\, D_x u$

(3) $D_x \tanh u = \text{sech}^2 u \,.\, D_x u$

(4) $D_x \coth u = -\text{csch}^2 u \,.\, D_x u$

(5) $D_x \text{sech}\, u = -\text{sech}\, u \,.\, \tanh u \,.\, D_x u$

(6) $D_x \text{csch}\, u = -\text{csch}\, u \,.\, \coth u \,.\, D_x u$

***Example* : 2**

Find $y^/$ for each of the following functions:

(a) $y = \cosh(x^2+1)$, (b) $y = \sqrt{\text{sech}\,5x}$, (c) $y = \sqrt{x}\tanh\sqrt{x}$

***Solution* :**

(a) If : $y = \cosh(x^2+1)$

§§§. $= 2x\sinh(x^2+1)$ $y^/ = \sinh\underbrace{(x^2+1)}_{u}\, D_x(x^2+1)$

(b) If : $y = \sqrt{\text{sech}\,5x}$ → $y = (\text{sech}\,5x)^{1/2}$

$$y^/ = \frac{1}{2}(\text{sech}\,5x)^{-1/2}.[-\text{sech}\underbrace{5x}_{u}.\tanh 5x\; D_x(5x)]$$

$$= -\frac{5}{2\sqrt{|\,\text{sech}\,5x\,|}}.|\,\text{sech}\,5x\,|.\tanh 5x = -\frac{5}{2}.\sqrt{|\,\text{sech}\,5x\,|}.\tanh 5x$$.§§§ .

(c) If : $y = \sqrt{x}\tanh\sqrt{x}$ → $y = \underbrace{\sqrt{x}}_{f}\;\underbrace{\tanh\sqrt{x}}_{g}$

Using a differential obtained by multiplying two functions:

$$y^/ = \underbrace{\sqrt{x}}_{f}.\underbrace{\left\{\text{sech}^2\sqrt{x}\,.D_x(\sqrt{x})\right\}}_{g^/} + \underbrace{\left\{\frac{1}{2\sqrt{x}}\right\}}_{f^/}.\underbrace{\tanh x^{1/2}}_{g}$$

$$= \sqrt{x}.\left\{\text{sech}^2\sqrt{x}.\frac{1}{2\sqrt{x}}\right\} + \left\{\frac{1}{2\sqrt{x}}\right\}.\tanh\sqrt{x}$$

$$= \frac{1}{2}\text{sech}^2\sqrt{x} + \frac{1}{2\sqrt{x}}.\tanh\sqrt{x}$$.§§§

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§§§§§§ **Try by yourself** §§§§§§

I- Find y^l for the following functions

(1) $y = \tan^{-1}(\text{csch}\,x)$, (2) $y = \ln|\tanh 3x|$, (3) $y = \coth\,\ln x$

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III-3 Inverse Hyperbolic Functions

The hyperbolic function $\sinh x$ is a continuous and increasing function of all values of continuous and increasing . Therefore, there is an inverse function of this function that is x symbolized by $\sinh^{-1} x$ .Since the function , from this we e^x can be expressed as $\sinh x$ should expect that the inverse function $\sinh^{-1} x$ can be expressed as $\ln x$ an inverse function of relations as shown in the following e^x

III-1 Inverse Hyperbolic Functions Representation

(1) $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$

(2) $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$, $x \geq 1$

(3) $\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x}$, $|x| < 1$

(4) $\operatorname{sech}^{-1} x = \ln \frac{x + \sqrt{1-x^2}}{x}$, $0 < x \leq 1$

III-2 Derivative Rules For Inverse Hyperbolic Functions

If $u = g(x)$ is a differentiable function , then

(1) $D_x \sinh^{-1} u = \frac{1}{\sqrt{u^2+1}} . D_x u$

(2) $D_x \cosh^{-1} u = \frac{1}{\sqrt{u^2-1}} . D_x u$, $u > 1$

(3) $D_x \tanh^{-1} u = \frac{1}{1-u^2} . D_x u$, $|u| < 1$

(4) $D_x \operatorname{sech}^{-1} u = \frac{-1}{u\sqrt{1-u^2}} . D_x u$, $0 < u < 1$

Example : 3

Find $y^{/}$ for each of the following functions :

(a) $y = \sinh^{-1}(\tan x)$, (b) $y = \text{sech}^{-1}(x^2)$

Solution :

(a) If : $y = \sinh^{-1}(\tan x)$

$$y = \sinh^{-1}\underbrace{(\tan x)}_{u} \Rightarrow y^{/} = \frac{1}{\sqrt{(\tan x)^2 + 1}} D_x(\tan x)$$

$$y^{/} = \frac{1}{\sqrt{\sec^2 x}} .\sec^2 x = \frac{1}{|\sec x|} .|\sec x|^2 = |\sec x|$$.§§§

(b) If : $y = \text{sech}^{-1}(x^2)$

$$y = \text{sech}^{-1}\underbrace{(x^2)}_{u} \Rightarrow y^{/} = \frac{-1}{x^2\sqrt{1-(x^2)^2}} D_x(x^2)$$

§§§. $$= \frac{-2}{x\sqrt{1-x^4}} \quad y^{/} = \frac{-2x}{x^2\sqrt{1-x^4}}$$

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§§§§§§ **Try by yourself**

I- Find y^{l} for the following functions

(1) $y = x\sinh^{-1}(\frac{1}{x})$, (2) $y = \frac{1}{\text{sech}^{-1}x^2}$

(3) $y = \ln\cosh^{-1}4x$, (4) $y = \tanh^{-1}\sin 3x$

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Exercise (5-2)

Hyperbolic and inverse hyperbolic

I- Find y^l for the following functions

(1) $y = \sinh^{-1} 5x$, (2) $y = \text{sech}^{-1}\sqrt{1-x}$

(3) $y = \tanh^{-1} x^3$, (4) $y = (\text{sech}^{-1} x)^{-1}$

(5) $y = \tanh^{-1}(\tanh \sqrt[3]{x})$, (6) $y = \cosh^{-1}(\tan x)$

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CHAPTER 6

INTEGRALS AND INTEGRALS APPLICATIONS

I- INTEGRALS

One of the great achievements of classical geometry was to obtain formulas for the areas and volumes of triangles, spheres, and cones. The method we develop, called *integration*, is a tool for calculating much more than areas and volumes. The *integral* has many applications in statistics, economics, the sciences, and engineering. The idea behind integration is that we can effectively compute many quantities by breaking them into small pieces, and then summing the contributions from each small part. We develop the theory of the integral in the setting of area, where it most clearly reveals its nature.

Definition : *(6.1)*

A function *F* is an **Anti-derivative** of the function of the function f on an interval *I* if $F^{/}(x) = f(x)$ for every *x* in *I*.

We shall also call *F(x)* an anti-derivative of *f(x)*. The process of finding *F* or *F(x)*, is called anti-differentiation.

e.g. :

$F(x) = x^2$ is an anti-derivative of $f(x) = 2x$, because $F^{/}(x) = \frac{d}{dx}(x^2) = 2x = f(x)$. there are many other derivative of *2x*, such as $x^2 + 2$, $x^2 - \frac{5}{3}$ and $x^2 + \sqrt{3}$. In general if *C* is any constant, then $x^2 + C$ is an anti-derivative of $2x$ as : $\frac{d}{dx}(x^2 + C) = 2x + 0 = 2x.$

I-1 INDEFINITE INTEGRALS

Definition : *(6.2)*

The notation $\int f(x)dx = F(x) + C$ where $F^{/}(x) = f(x)$ and C is an arbitrary constant, denotes the family of all anti-derivative of $f(x)$ on an interval I.

e.g. :

- ► $\int x^4 dx = \frac{1}{5}x^5 + C$ because $\frac{d}{dx}(\frac{1}{5}x^5) = x^4$. §§§
- ► $\int t^{-3} dt = \frac{-1}{2}t^{-2} + C$ because $\frac{d}{dx}(\frac{-1}{2}t^{-2}) = t^{-3}$. §§§
- ► $\int \cos u du = \sin u + C$ because $\frac{d}{du}(\sin u) = \cos u$. §§§

Indefinite Integrals Forms

Derivative $\frac{d}{dx}(f(x))$	**Indefinite integral** $\int \frac{d}{dx}(f(x))dx = f(x) + C$
$\frac{d}{dx}(x) = 1$	$\int 1 dx = \int dx = x + C$
$\frac{d}{dx}(\frac{x^{r+1}}{r+1}) = x^r ; (r \neq -1)$	$\int x^r dx = \frac{x^{r+1}}{r+1} + C \;\; ; r \neq -1$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\, dx = \sin x + C$
$\frac{d}{dx}(-\cos x) = -\sin x$	$\int \sin x\, dx = -\cos x + C$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\, dx = \tan x + C$
$\frac{d}{dx}(-\cot x) = \mathrm{cosec}^2 x$	$\int \csc^2 x\, dx = -\cot x + C$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x\, dx = \sec x + C$
$\frac{d}{dx}(-\mathrm{cosec} x) = \mathrm{cosec} x \cot x$	$\int \csc x \cot x\, dx = -\csc x + C$

e.g. :

- ► $\int x^3 . x^5 dx = \int x^8 dx = \frac{x^9}{9} + C$. §§§ , ► $\int \frac{1}{x^3} dx = \int x^{-3} dx = -\frac{x^{-2}}{2} + C$. §§§
- ► $\int \frac{1}{x^3} dx = \int x^{-3} dx = -\frac{x^{-2}}{2} + C$. §§§ , ► $\int \sqrt[3]{x^2} dx = \int x^{\frac{2}{3}} dx = \frac{3}{5} x^{\frac{5}{3}} + C$. §§§
- ► $\int \frac{\tan x}{\sin x} dx = \int \cos x \frac{\sin x}{\cos x} dx = \int \sin x dx = -\cos x + C$. §§§

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Theorem **: (6-1)**

1) $\frac{d}{dx}(\int(f(x))dx = f(x)$ **2)** , $\int\frac{d}{dx}(f(x))dx = f(x)+C$

e.g.

If $f(x) = x^2$

* If we first integrate x^2 and then integrate, $f(x) = x^2$

$$\int\frac{d}{dx}(x^2)dx = \int 2xdx = x^2 + C \cdot$$ §§§

** If we first integrate x^2 and then differentiate,

$$\frac{d}{dx}(\int(x^2)dx) = \frac{d}{dx}(\frac{x^3}{3}+C) = x^2 \cdot$$ §§§

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I-2 Rules Of Indefinite Integrals

The next theorem is useful for evaluating many types of the indefinite integrals. In the statements, we assume that $f(x)$ and $g(x)$ have anti-derivatives on an interval *I*.

Theorem **: (6-2)**

1) . k constant for any nonzero $\int kf(x)dx = k\int f(x)dx$

2) $\int[f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

3) $\int(f(x))^n f^{/}(x)dx = \frac{(f(x))^{n+1}}{n+1} + C$, $\forall n \neq -1$

4) $\int[f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$, $\forall n \neq -1$

e.g. :

► $\int 3\sin xdx = 3\int \sin xdx = -3\cos x + C$. §§§

► $\int e\sec^2 xdx = e\int \sec^2 xdx = e\tan x + C$. §§§

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***Example* : 1**

Evaluate each of the following integrals :

► $\int(5x^3+2\cos x)dx=5\int x^3dx+2\int\cos xdx\ \frac{5}{4}x^4+2\sin x+C$.

► $\int(\sqrt{x}+\frac{1}{\sqrt{x}}+\cos x-\operatorname{cosec}^2 x)dx$

$=\int\sqrt{x}dx+\int\frac{1}{\sqrt{x}}dx+\int\cos xdx-\int\operatorname{cosec}^2 xdx$

§§§ $.=\frac{2}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}+\sin x+\cot x+C$

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► $.\int(\frac{1}{2\sqrt{x}}-3\cos x+2\sec x\tan x)dx=\sqrt{x}-3\sin x+2\sec x+C$

► $\int(8t^3-6\sqrt{t}+\frac{1}{t^3})dx=\frac{8}{4}t^4-6.\frac{2}{3}t^{\frac{3}{2}}+\frac{t^{-2}}{-2}+C$

§§§ $.=2t^4-4t^{\frac{3}{2}}-\frac{1}{2t^2}+C$

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***Example* : 2**

Evaluate each of the following integrals :

► $\int(\sqrt{x^2+1}.(2x))dx=\frac{2}{3}(x^2+1)^{\frac{3}{2}}+C.$

► $\int\frac{\sin 2x}{\sqrt{1+\sin^2 x}}dx=\int(1+\sin^2 x)^{\frac{-1}{2}}.2\sin x\cos xdx$

§§§ $.=2\sqrt{1+\sin^2 x}+C$

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***Example* : 3**

Evaluate each of the following integrals :

► $=\frac{1}{3}x^3-2x-\frac{1}{x}+C\ \int\frac{(x^2-1)^2}{x^2}dx=\int\frac{x^4-2x^2+1}{x^2}dx=\int(x^2-2+x^{-2})dx$.

► $\int\frac{1}{\cos u\cot u}du=\int\sec u\tan udu=\sec u+C.$

***Example* : 4**

Evaluate each of the following integrals :

► $\int\frac{\cos\sqrt{x}}{\sqrt{x}}dx=2\int\cos\sqrt{x}.\frac{1}{2\sqrt{x}}dx=2\sin\sqrt{x}+C.$

►§§§ $.\int x\sec(x^2)\tan(x^2)dx=\frac{1}{2}\int\sec(x^2)\tan(x^2).2xdx=\frac{1}{2}\sec(x^2)+C$

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***Example* : 5**

Evaluate the integral : $I = \int x^2 \sqrt{x^3 + 9}dx$

***Solution* :**

Use suitable substitution for the given integral as :

Put $u = x^3 + 9$ then $du = 3x^2dx$,

Hence:

$$\int x^2 \sqrt{x^3 + 9}dx = \frac{1}{3}\int \sqrt{u}du = \frac{1}{3}.\frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{9}(x^3 + 9)^{\frac{3}{2}} + C .$$ §§§

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REMEMBER

Some of trigonometric relations play an important role in calculation of the **integration process of squares trigonometric functions** as :

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad , \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$1 + \tan^2 x = \sec^2 x \quad , \quad 1 + \cot^2 x = \csc^2 x$$

***Example* : 6**

Evaluate each of the following integrals :

► $\int \sin^2 x dx = \int \frac{1}{2}(1 - \cos 2x)dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C .$ §§§

► $\int \cos^2 x dx = \int \frac{1}{2}(1 + \cos 2x)dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C .$ §§§

► $\int \tan^2 x dx = \int (\sec^2 x - 1)dx = \tan x - x + C .$ §§§

► $\int \cot^2 x dx = \int (\operatorname{cosec}^2 x - 1)dx = -\cot x - x + C .$ §§§

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***Example* :7**

Evaluate each of the following integrals :

► $\int \frac{2 - \cos^2 x}{\cos^2 x} dx = \int (2\sec^2 x - 1)dx = 2\tan x - x + C .$ §§§

► $\int \frac{x}{\sqrt{x - 1}} dx = \int \frac{x - 1 + 1}{\sqrt{x - 1}} dx = \int \sqrt{x - 1} + \frac{1}{\sqrt{x - 1}}dx$

§§§ $. = \frac{2}{3}(x - 1)^{\frac{3}{2}} + 2(x - 1)^{\frac{1}{2}} + C$

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Exercise *(6-1)*

Evaluate each of the following integrals:

1) $\int (3x^5 + 5x^2 - 7)dx.$, 2) $\int (\frac{3}{x^3} + 2x^{\frac{3}{2}} - 4)dx$

3) $\int (3x - 2)^5 dx.$, 4) $\int \cos(3x - 1)dx.$

5) $\int (x^3 \sqrt{1 - x^2})dx.$, 6) $\int x \cos x^2 dx.$

7) $\int \frac{\tan^2 \sqrt{x}}{\sqrt{x}} dx.$, 8) $\int (\sin^{\frac{3}{2}} 2x + 8x)dx.$

9) $\int (25\pi + 17x^2)dx.$, 10) $\int [\tan^2(3x) - \sin x]dx$

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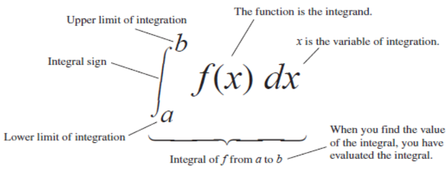
I-3 Definite Integrals

Let $F(x)$ be an anti-derivative of $f(x)$. Then we define the definite integral of $f(x)$ with respect to x between the limits $x = a, x = b$ (i.e. in the closed interval [a, b]) as:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$



***Example* :8**

Evaluate each of the following integrals:

► $\int_0^1 \sqrt{x} - x^2 dx = \left[\frac{x^{\frac{3}{2}}}{3/2} - \frac{x^3}{3} \right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$ §§§

► $\int_0^{\frac{\pi}{2}} \sin^2 x \cos x dx = \frac{1}{3}\left[\sin^3 x\right]_0^{\frac{\pi}{2}} = \frac{1}{3}(1-0) = \frac{1}{3}.$ §§§

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I-4 Properties Of Definite Integrals

Definite integrals have some interesting properties which we will discuss them in the following theorem:

If f and g are integrable on [a, b], then $f \pm g$ is integrable on [a, b] and:

(1) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$ **for any nonzero constant *k* .**

(2) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$

(3) $\int_a^a f(x)dx = 0.$, **4)** $\int_b^a f(x)dx = -\int_a^b f(x)dx.$

5) **For** $c \in [a,b], \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$

***Example* :9**

Evaluate each of the following integrals :

► $\int_1^4 (\sqrt{x} + \frac{1}{\sqrt{x}})dx = \frac{2}{3}\left[x^{\frac{3}{2}}\right]_1^4 + 2\left[x^{\frac{1}{2}}\right]_1^4 = \frac{20}{3}$ §§§

► $\int_0^4 (7x^{\frac{5}{2}} - 5x^{\frac{3}{2}})dx = 7.\frac{2}{7}\left[x^{\frac{7}{2}}\right]_0^4 - 5.\frac{2}{5}\left[x^{\frac{5}{2}}\right]_0^4 = 256 - 64 = 192$. §§§

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***Example* :10**

Evaluate $\int_0^3 (x^2\sqrt{x^3+9})dx$.

Solution

Using integration by substitution: Let $u = x^3 + 9 \Rightarrow du = 3x^2 dx$

Since $x = 0 \Rightarrow u = 9$ and $x = 3 \Rightarrow u = 36$

Thus, $\int_0^3 (x^2\sqrt{x^3+9})dx = \frac{1}{3}\int_9^{36} u^{\frac{1}{2}}du = \frac{1}{3}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_9^{36} = 42.$ §§§

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***Example* :10**

Evaluate $\int_0^{\frac{\pi}{4}} \tan^2\theta \sec^2\theta dx$.

Solution

Using integration by substitution: Let $u = \tan\theta \Rightarrow du = \sec^2\theta d\theta$

Since $\theta = 0 \Rightarrow u = 0$ and $\theta = \frac{\pi}{4} \Rightarrow u = 1$

Thus, $\int_0^{\frac{\pi}{4}} \tan^2\theta \sec^2\theta dx = \int_0^1 u^2 du = \frac{1}{3}\left[u^3\right]_0^1 = \frac{1}{3}.$

An important property in definite integral:

$$\int_{-a}^{a} f(x)dx = \begin{cases} 0, & \text{if } f(x) \text{ is odd on } [-a,a] \\ 2\int_0^a f(x)dx & \text{if } f(x) \text{ is even on } [-a,a] \end{cases}.$$ §§§

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***Example* :11**

Evaluate each of the following integrals :

► $\int_{-2}^{2} x\sqrt{x^2+1}\ dx = 0$, as the integrand is an odd function of x . §§§

► $\int_{-2}^{2} x^2(x^2+1)\ dx = 2\int_0^2 x^2(x^2+1)\ dx = 2\left[\frac{x^5}{5} + \frac{x^3}{3}\right]_0^2 = 2\left[\frac{32}{5} + \frac{8}{3}\right] = \frac{272}{15}$.

The integrand here is an even function of *x*. §§§

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Exercise (6-2)

Evaluate the following integrals:

(1) $\int_{-2}^{3} x(x^2+1)dx$. (2) $\int_{-1}^{2} x^2(x^3-1)^{\frac{1}{2}}dx.$

(3) $\int_{1}^{3} \frac{x^3}{\sqrt{x^2+1}}dx.$. (4) $\int_{0}^{\frac{\pi}{4}} \sec^2 x\sqrt{1+\tan x}dx$

(5) $\int_{0}^{15} R\sqrt{R+1}dR.$. (6) $\int_{0}^{\frac{\pi}{4}} \theta\sin(\frac{\theta^2}{2})d\theta.$

(7) $\int_{0}^{\frac{\pi}{2}} (1+\sin t)^{\frac{3}{2}}\cos t dt$. (8) $\int_{0}^{\frac{\pi}{4}} \frac{\sin x}{\sqrt{\cos x}}dx$

(9) $\int_{-1}^{1} \frac{x+1}{(x^2+2x+2)^{\frac{1}{2}}}dx$. (10) $\int_{0}^{\frac{\pi}{2}} \frac{\sin 2x}{\sqrt{4\sin^2 x+3\cos^2 x}}dx$

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II- DEFINITE INTEGRALS APPLICATIONS

This chapter discuss some of ***Definite Integral*** uses , firstly we start by applications explain how can be use to calculate volume (for irregular shapes) as the regular such as , cubes , cylinders , spheres , ….etc has its own algebraic rules to calculate easily , and we also discuss how to find arc-length for certain curve, surface area of solid bodies(specially irregular bodies)

Let us remember that primal integral definition state that :

" ***Definite integral is a limit of sum*** "

II-1 Area Of A Region

This section help us to calculate irregular area lies between two graphs or more .The following theorem descript how to find this area :

Theorem **:6-3**

If f, g are two continuous functions as $f(x) \ge g(x)$ for eachand the two $x \in [a,b]$ straight line $x = a$, $x = b$ then the bounded resulted area (Say) of intersection can be given by :

$$A = \int_a^b [\, f(x) - g(x)]\, dx$$

fig. (1) describe the theorem .

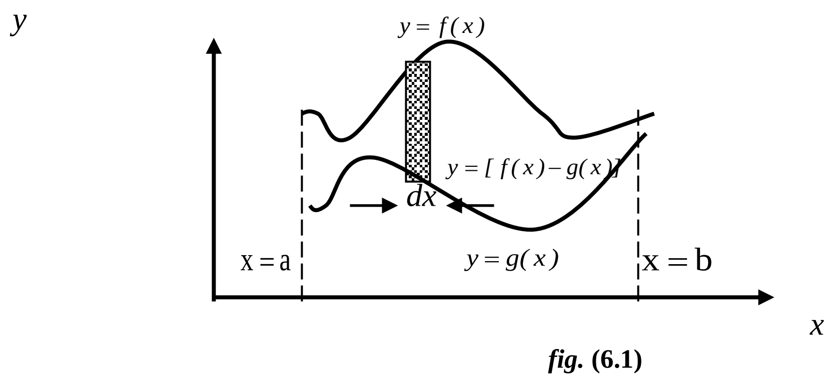

***fig.* (6.1)**

The integral in theorem (6-1) represented by fig.(1.1) by the shadow rectangular slide as :

$$A = \int_{x=a}^{b} \underbrace{[f(x) - g(x)]}\; dx$$

Integral boundaries   Rectangle length   Rectangle width

Guide steps for calculating Bounded area :

*** Draw the required area bounded from above by the graph* $y = f(x)$ *and Bounded from below by the graph* $y = g(x)$ *and determine the upper and lower Integral boundaries .*

*** Find the rectangular shadow area with width dx and length* $y = [f(x) - g(x)]$ *, and then the slide area represented.* $dA = [f(x) - g(x)].dx$

** Finally the total area calculated by :* $A = \int_a^b dA = \int_a^b [f(x) - g(x)].dx$.

Example :12

Evaluate the area bounded by the curves $y = x^2$, $y = \sqrt{x}$.

Solution:

By using ***guide steps*** :

Upper graph : $y = \sqrt{x}$
Lower graph $y = x^2$:
Slide width : dx
Slide length : $(\sqrt{x} - x^2)$
Slide area : $dA = (\sqrt{x} - x^2)\,dx$

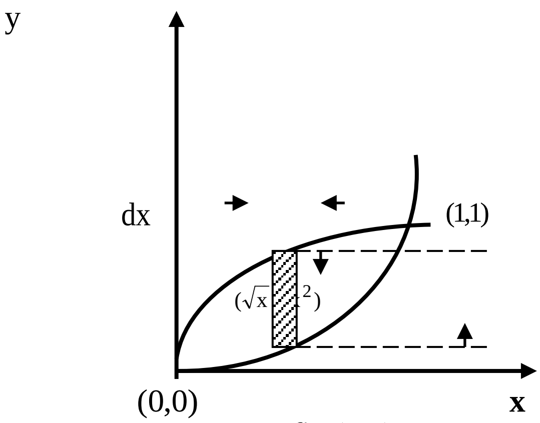

***fig.* (6.2)**

The guide steps describe fig. (1.2)

To calculate the integral value (i.e. the area) apply the following steps:

** Solve the equations of the two graphs $y = \sqrt{x}$, $y = x^2$ together to get the points of intersect which are $(0, 0)$ and $(1, 1)$.

** The integral <u>boundaries</u> are . $x : 0 \to 1$

** <u>The area</u> A determined by :

$$A = \int_0^1 dA = \int_0^1 (\sqrt{x} - x^2)\,dx = \int_0^1 (x^{1/2} - x^2)\,dx$$

$$= \left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right]_0^1 = \left[\frac{2}{3} - \frac{1}{3}\right] = \frac{1}{3} \quad . §§$$

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Example : 13

Evaluate the area bounded by the curves : $y+2x-3=0$, $y+x^2=6$.

Solution:

By using ***guide steps***:

Upper graph : $y=6-x^2$
Lower graph $y=3-2x$:
Slide width : dx
Slide length : $[(6-x^2)-(3-2x)]$
Slide area : $dA=[(6-x^2)-(3-2x)]dx$

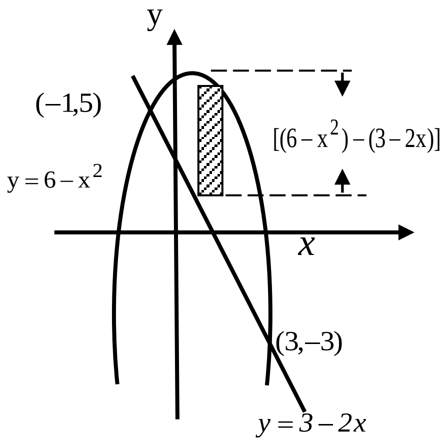

***fig.* (6.3)**

The guide steps describe fig. (6.3)

To calculate the integral value (i.e. the area) apply the following steps:

** Solve the equations of the two graphs $y=6-x^2, y=3-2x$ together to get the points of intersect which are $(-1,5)$ and $(3,-3)$.

** The integral boundaries are . $x:-1\to 3$

** The area A determined by :

$$A=\int_{-1}^{3} dA=\int_{-1}^{3}[(6-x^2)-(3-2x)].\,dx$$

$$=\int_{-1}^{3}(3-x^2+2x).\,dx=\left[3x-\frac{x^3}{3}+x^2\right]_{-1}^{3}$$

$$=\left(9-\frac{27}{3}+9\right)-\left(-3-(-\frac{1}{3})+1\right)=\frac{32}{3}$$. §§

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***Example* : 14**

Evaluate the region R bounded by the curves $y-x=6$, $y-x^3=0$.

Solution:

By drawing the graphs we note that the region R has different lower bounds , so to find the required area of the region to two region R_1 , R_2
as in the following drawing , say $R=R_1+R_2$

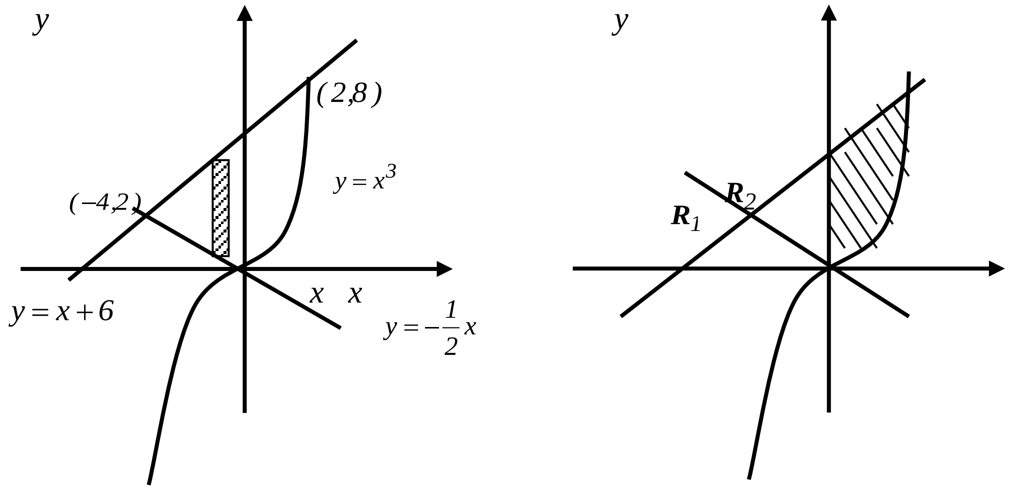

***fig.* (6.4)**

Region R_1
Upper graph : $y=x+6$
Lower graph $y=x^3$:
Slide width : dx
Slide length : $[(x+6)-x^3]$
Slide area $dA=[(x+6)-x^3].dx$:

Region R_2
Upper graph : $y=x+6$
Lower graph $y=-(1/2)x$:
Slide width : dx
Slide length : $[(x+6)-(1/2)x]$
Slide area : $dA=[(x+6)-(-1/x)x]dx$

The guide steps describe fig.(6.4)

** To find points of intersection for each area we solve the equations the two equations $y=x+6$, $y=-\frac{1}{2}x$ together with respect to y we get the point $(-4,2)$, and in similar way we solve the equations $y=x+6$, $y=x^3$ together to get the point (2,8)

** Then calculate the districted two area R_1 , R_2 we get :

$$A_1 = \int_{R_1} dA_1 = \int_{-4}^{0} [(x+6)-(-\frac{1}{2}x)].dx$$

$$= \int_{-4}^{0} \left(\frac{3}{2}x+6\right).dx = \left[\frac{3}{2}(\frac{x^2}{2})+6x\right]_{-4}^{0}$$

$$= 0-(12-24) = 12 \quad . §$$

$$A_2 = \int_{R_2} dA_1 = \int_{0}^{2} [(x+6)-x^3].dx$$

$$= \left[\frac{x^2}{2}+6x-\frac{x^4}{4}\right]_{0}^{2} = (2+12-4)-0 = 10 \ . §$$

Then the required area A is given as :

$$A = A_1 + A_2 = 12+10 = 22 \ . §§$$

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NOTE

** the integrals in examples 1,2 and 3 above the slides are vertical Rectangle moved horizontal from left to right as this area bounded by continuous functions $y = f(x)$ and $y = g(x)$ proposed with $f(x) \geq g(x)$ which we denoted symbolic by R_x .

** In some integral case these proposion may be complicate the integral , so we need to get another slide selection , and the horizontal selection for slide in this case is suitable to simplify the integral as we see in the next example 4, we denoted symbolic by R_y .

***Example* : 15**

Evaluate the region R bounded by the curves $2y^2 = x + 4$, $y^2 = x$.

Solution:

To simplify the equation drawing we satisfy the given equations to take the form $x = f(y)$ and $x = g(y)$ which leads to equivalent forms $x = 2y^2 - 4$ and $x = y^2$.

Note that the choose for vertical slide as in fig. (6.5-a) (i.e. integrate with respect to x) leads to evaluate three different integrals . In such case choose horizontal slide as in fig. (6.5-b) and then integrate with respect to y .

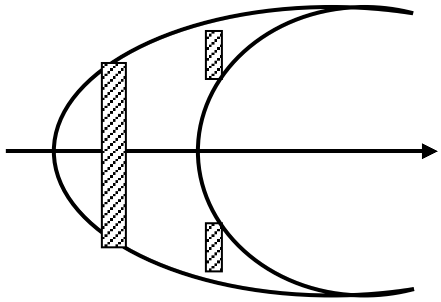

fig. (6.5a)

Right graph : $x = y^2$

Left graph $x = 2y^2 - 4$:

Slide width : dy

Slide length : $[y^2 - (2y^2 - 4)]$

Slide area : $[dA = y^2 - (2y^2 - 4)].dy$

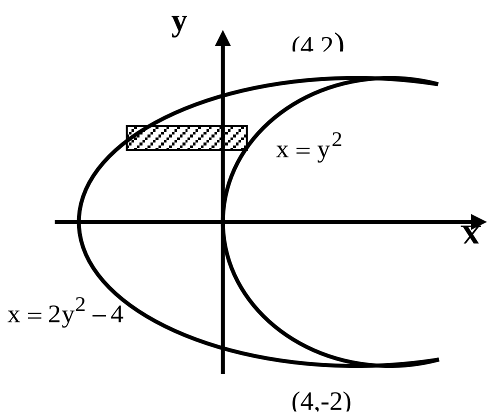

fig. (6.5b)

Then the required area A is given as :

$$A = \int_{y=-2}^{2} dA = \int_{y=-2}^{2} [y^2 - (2y^2 - 4)]dy$$

$$= 2\int_{0}^{2} [4 - y^2]dy = 2\left[4y - \frac{y^3}{3}\right]_0^2$$

$$= 2\left(8 - \frac{8}{3}\right) = \frac{32}{3}$$. §§

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Exercise (6-3)

Bounded Area

Draw and calculate the area bounded by each of the following graphs :

(1) $y = x^2$; $y = 4x$, (2) $x + y = 3$; $y + x^2 = 3$

(3) $y = 4 - x^2$; $y = -4$, (4) $x = y^2$; $y - y = 2$

(5) $y = x^2 + 1$; $x - y = 2$, (6) $y = \sqrt{x}$; $x + y = 6$; $x = 1$

(7) $x = -3y^2 + 4$; $x = y^2$, (8) $y = 3x$; $y = x$; $x + y = 4$

(9) $y = x^3$; $y = x^2$, (10) $y^2 = 4 + x$; $y^2 + x = 2$

(11) $y^2 = -x$; $x - y = 4$; $y = -1$; $y = 2$.

(12) $x = y^2$; $y - x = 2$; $y = -2$; $y = 3$.

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II-2 Solid Of Revolution

If we need to find the center of mass or the moment of inertia for irregular solid body we need first to calculate the volume of that body which doesn't apply the algebraic rules for regular bodies . ***Definite integral*** is one of the most important tools used for calculate such volumes .

** Volumes arise from ***revolving area*** (region) in plane ***around certain axis*** or ***certain straight line*** .

** The arising volume resulting in such operation is called ***the Solid of Revolution*** .

** The area arise this volume is called the arising area .

** The axis (or straight line) which revolved around it is called ***the axis of revolution*** .

for example if the region R_x fig. (6.6-a) revolved around the x-axis as an axis of revolution , then the resulted solid volume arising is fig. (6.6-b)

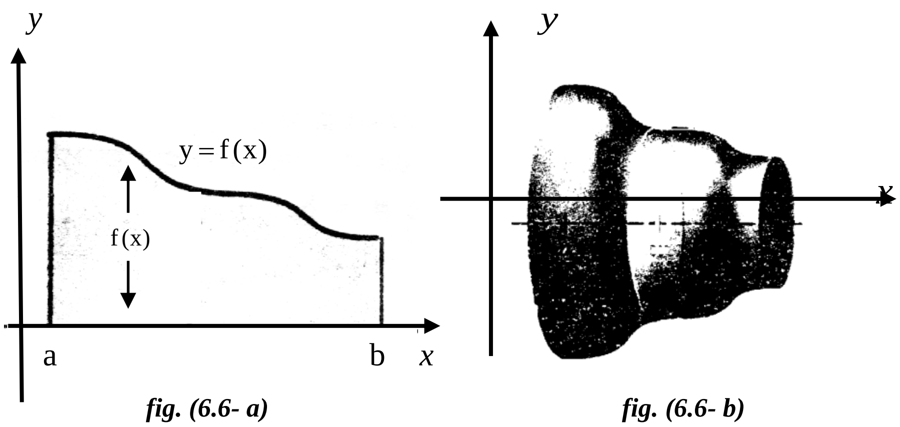


fig. (6.6- a) ***fig. (6.6- b)***

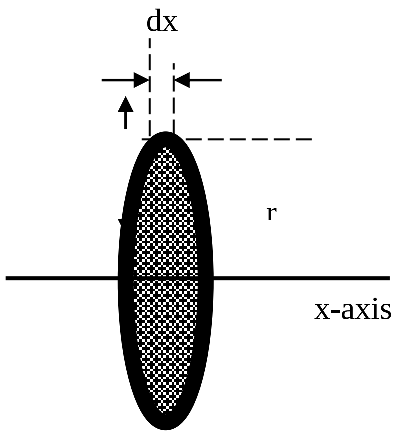
As the revolved region can be revolve around x-axis, y-axis , or any straight line in plane , so its impossible to formulate one definite form for integral . So its very useful to remember ***the volume of circular solid disk*** which represent in that case the element volume dV for the resulted solid volume of revolution .

NOTE

** Revolve a slide on rectangle shape resulting a ***circular solid disk*** represent element volume dV for the resulted solid volume of revolution .

Volume OF Circular Solid Disk

Solid disk thickness : dx
Radius of revolution : r
Axis of revolution : x
Slide volume : $dV = \pi r^2 dx$



Guide steps for calculating Solid Of Revolution :

*** Draw the required **area need to revolve** and specify its boundaries.*

*** Find the rectangular vertical slide with thickness dx or the rectangular horizontal slide with thickness* dy *.*

*** Draw the volume which arising from the revolution of region R around the axis of revolution and determine the resulted circular disk .*

*** Represent about the radius of a disk as a function of x(or a function of y) according to the disk thickness* dx *(or dy) respectively .*

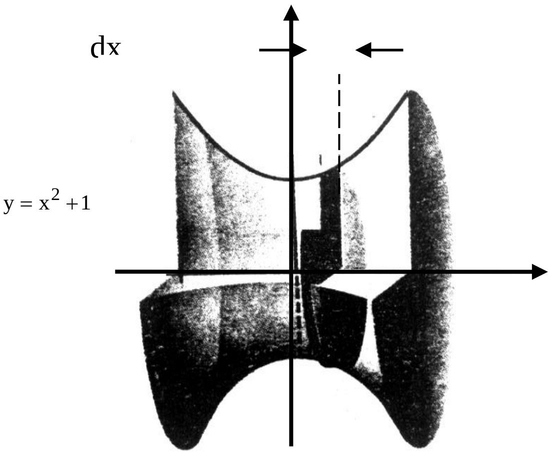
*** The resulting volume given by the integral :* $V = \int_a^b dV$

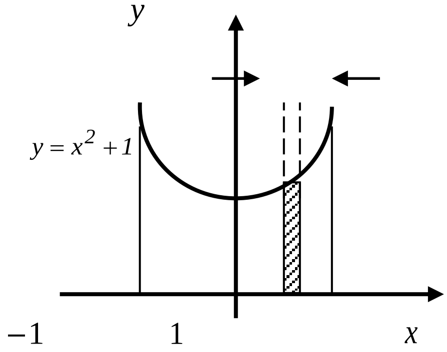
Example :16

Find the volume of revolution if the area bonded by x-axis , the graph , and the $y = x^2 + 1$ two straight lines $x = 1,\ x = -1$ revolved about x-axis .

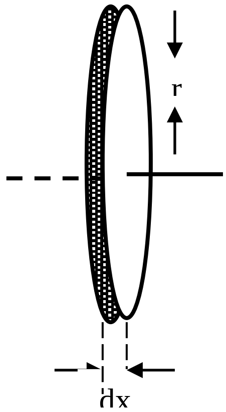
Solution:

Using the guide steps :





The solid revolving volume arise from revolving region R around axis of revolution x-axis , and by using guides steps



Solid disk thickness : dx
Radius of revolution : $r = (x^2 + 1)$
Axis of revolution : x
Slide volume : $dV = \pi(x^2 + 1)^2\, dx$

The resulting volume given by the integral :

$$V = \int_{-1}^{1} dV = \int_{-1}^{1} \pi(x^2 + 1)^2\, dx = 2\pi\int_0^1 (x^4 + 2x^2 + 1)^2\, dx$$

$$= 2\pi\left[\frac{x^5}{5} + 2.\frac{x^3}{3} + x\right]_0^1 = 2\pi\left[\frac{1}{5} + \frac{2}{3} + 1\right] = \frac{56}{15}\pi \;\cdot\; §§$$

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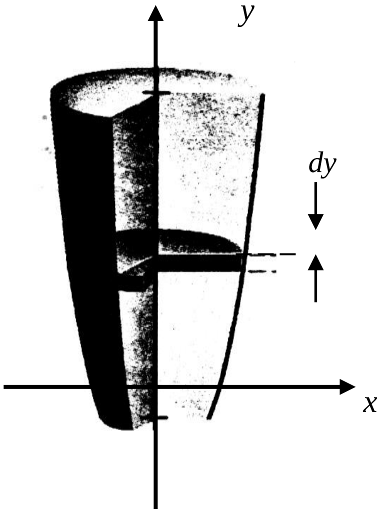
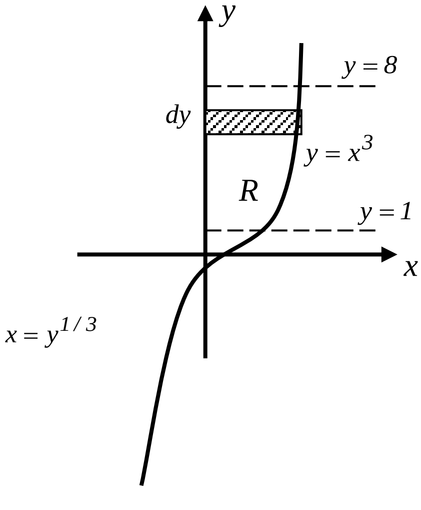
Example :17

Find the volume of revolution if the area bonded by y-axis , the graph , and the $y = x^3$ two straight lines $y = 1,\ y = 8$ revolved about y-axis .

Solution:

Using the guide steps :





The solid revolving volume arise from revolving region R around axis of revolution y-axis , and by using guides steps

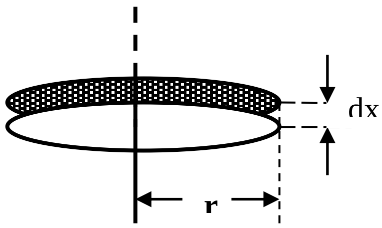
Volume OF Circular Solid Disk

Solid disk thickness : dy

Radius of revolution : $r = y^{1/3}$

Axis of revolution : y

Slide volume : $dV = \pi(y^{1/3})^2\,dy$



The resulting volume given by the integral :

$$V = \int_1^8 dV = \int_1^8 \pi(y^{1/3})^2\,dy = \pi\int_1^8 y^{2/3}\,dy$$

$$= \pi\left[\frac{y^{5/3}}{(5/3)}\right]_1^8 = \frac{3}{5}\pi[32-1] = \frac{93}{5}\pi \cdot$$

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Now , let the region R_x bounded by the two graphs $y = f(x)$ and $y = g(x)$ represented as in fig (6.7-a) supposing that it revolve about x-axis as an axis of revolution this leads to the solid volume of revolution fig (6.7-b)

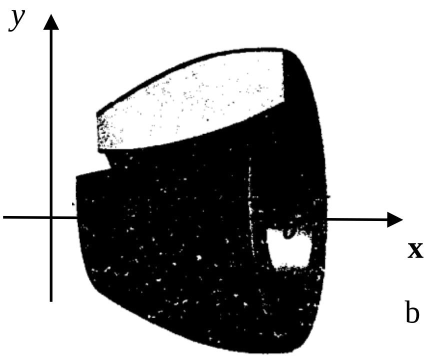


Fig . (6.7- b)

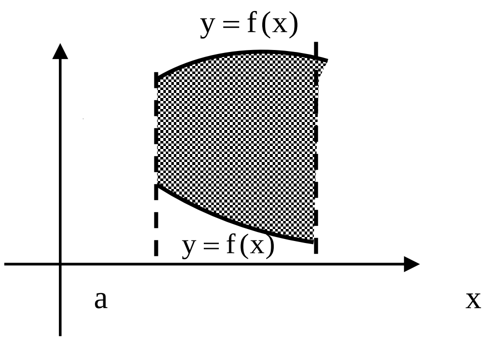


fig. (6.7- a)

The resulted solid volume of revolution contain an internal gap as $g(x) > 0$ and for each $x \in [a,b]$.

The volume of revolution V arises by subtracting the inner arising volume V_1 from the outer arising volume V_2 which can be represented by the relation :

$$V = (V_2 - V_1) = \int_a^b \pi[f(x)]^2\,dx - \int_a^b \pi[g(x)]^2\,dx$$

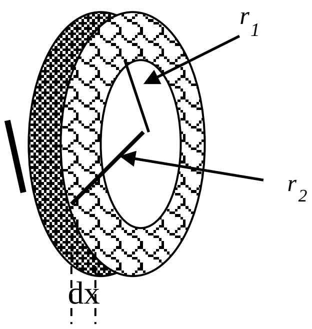
$$= \int_a^b \pi\left\{[f(x)]^2 - [g(x)]^2\right\}dx \qquad (*)$$

II-3 Washer Pattern

Integral (*) is called the ***Washer Pattern*** and can be mathematically formulated as :

$$V = \int_a^b \pi\left[r_2^2 - r_1^2\right].dx \qquad (**)$$

As :
Inner radius of revolution : r_1
Outer radius of revolution : r_2
Slide thickness : dx



and then the slide in this case has the fig. (6.7-c)

fig. (6.7-c)

NOTE

** when washer rule (**) an error can occurred if we use relation $(r_2 - r_1)^2$ instead of relation $(r_2^2 - r_1^2)$ in integral form .

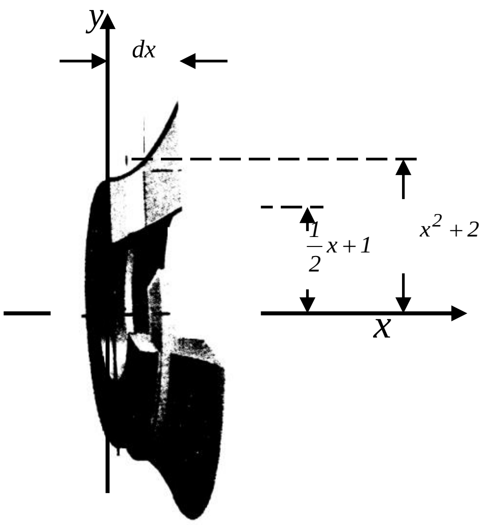
** Guide steps used in the solid volume ***are typically*** used in washer formula **but take in consideration** that the radius of revolution r is the difference square between the inner and outer radius i.e. $r = r_2^2 - r_1^2$.

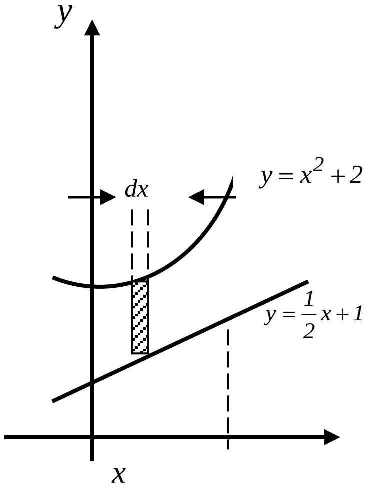
***Example* :18**

Find the volume of revolution if the area bonded by the graph , and the straight $x^2 = y - 2$ lines $2y - x - 2 = 0$ and the two vertical lines , $x = 0$, $x = 1$

Solution:

Using the guide step:



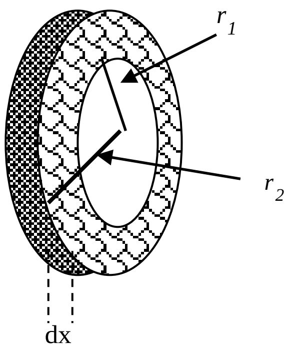


Inner radius: $r_1 = (\frac{1}{2}x + 1)$

Outer radius : $r_2 = (x^2 + 2)$

Slide thickness : dx

Slide volume : $dV = \pi\left[(x^2+2)^2 - (\frac{1}{2}x+1)^2\right]dx$



and then the slide in this case has the fig. (6.8)) ***(Washer Shape)*** ***fig. (6.8)***

$$V = \int_0^1 dV = \int_0^1 \pi\left[(x^2+2)^2 - (\frac{1}{2}x+1)^2\right]dx$$

$$= \pi\int_0^1 (x^4 + \frac{15}{4}x^2 - x + 3)\,dx$$

$$= \pi\left[\frac{x^5}{5} + \frac{15}{4}\left(\frac{x^3}{3}\right) - \frac{x^2}{2} + 3x\right]_0^1 = \frac{79}{20}\pi$$. §§

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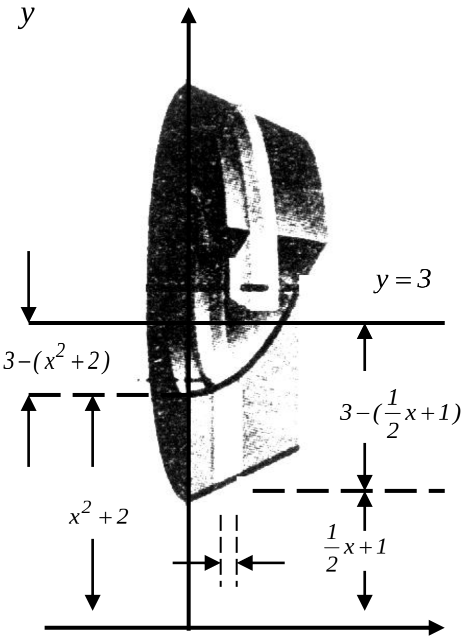
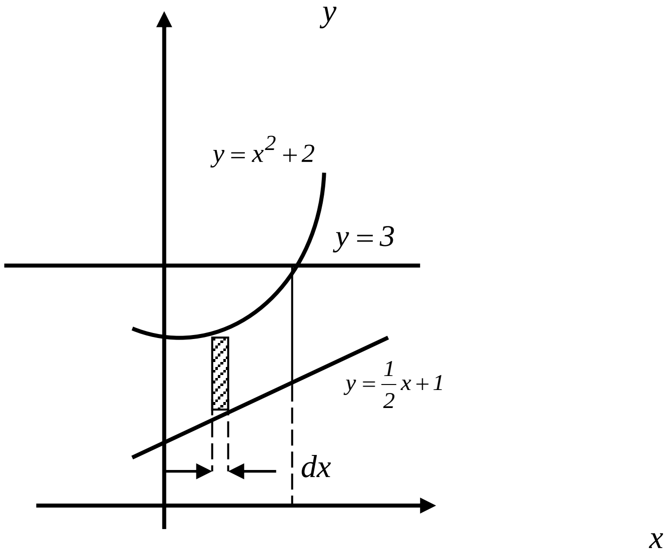
Example :19

Find the volume of revolution arise from revolving the region R of example (7) around the straight line $y = 3$ (axis of revolution) .

Solution:

Using the guide step:



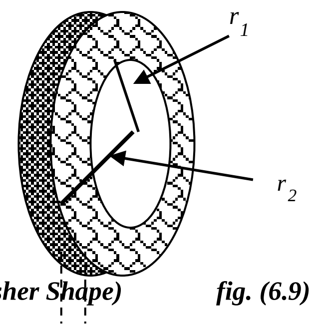


Inner radius: $r_1 = 3-(x^2+2) = 1-x^2$

Outer radius : $r_2 = 3-(\frac{1}{2}x+1) = 2-\frac{1}{2}x$

Slide thickness : dx

Slide volume : $dV = \pi\left[(2-\frac{1}{2}x)^2 - (1-x^2)^2\right]dx$



and then the slide in this case has the fig. (6.9) ***(Washer Shape)*** ***fig. (6.9)***

$$V = \int_0^1 dV = \int_0^1 \pi\left[(2-\frac{1}{2}x)^2 - (1-x^2)^2\right]dx$$

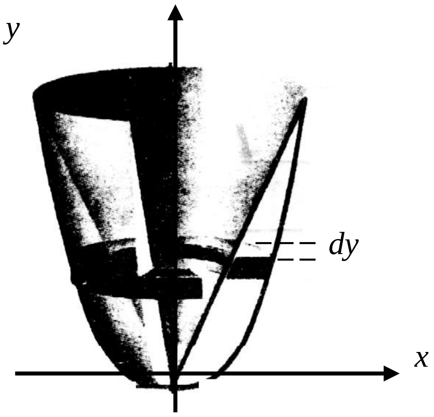
$$= \pi\int_0^1 (3-2x+\frac{9}{4}x^2 - x^4)\,dx = \pi\left[3x - x^2 + \frac{9}{4}\left(\frac{x^3}{3}\right) - \frac{x^5}{5}\right]_0^1 = \frac{51}{20}\pi \ . §§$$

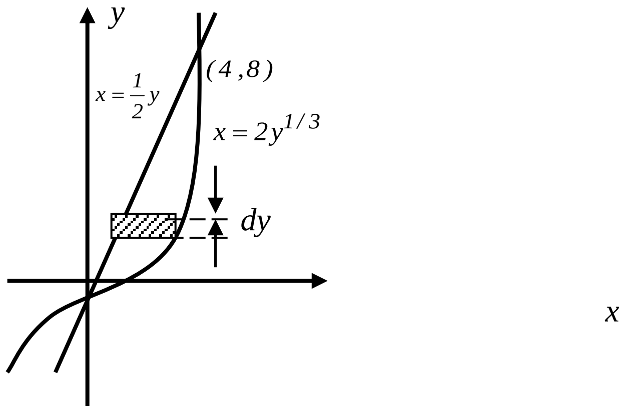
Example :20

Find the volume of revolution arise from revolving the region R about y-axis and lies in the 1st quadrant of coordinate , and bounded by the graph $y = (1/8)x^3$ and the straight line $y = 2x$.

Solution:

Using the guide step:





Its clear that a horizontal slide arise as in shape (fig. 10) For such revolution , so we must solve graph equation with respect to y and also the integral boundaries as follow:

Inner radius: $r_1 = (1/2)y$

Outer radius : $r_2 = 2y^{1/3}$

Slide thickness : dy

Slide volume : $dV = \pi\left[(2y^{1/3})^2 - (\frac{1}{2}y)^2\right]dy$

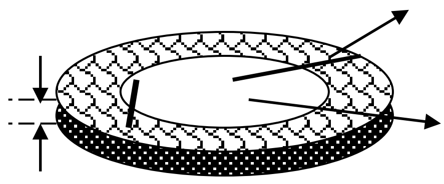


fig. (6.10)

and then the slide in this case has the fig. (10) ***(Washer Shape)***

$$V = \int_0^8 dV = \int_0^8 \pi(4y^{2/3} - \frac{1}{4}y^2)\,dy \quad = \pi\left[\frac{12}{5}y^{5/3} - \frac{1}{12}y^3\right]_0^8 = \frac{512}{15}\pi \ . §§$$

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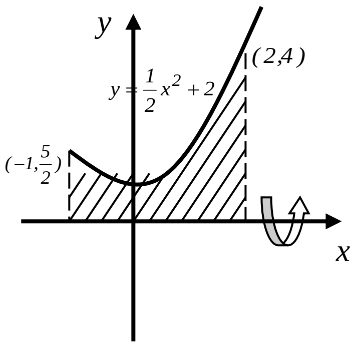
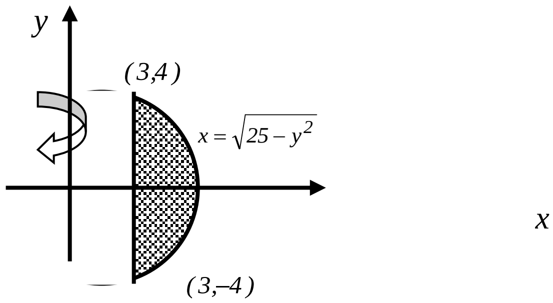
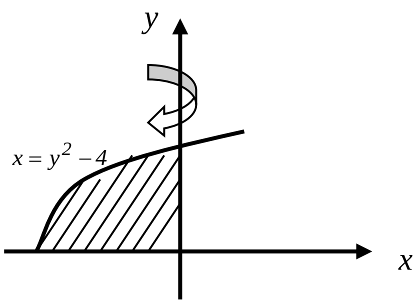
E x e r c i s e ***(6-4)***

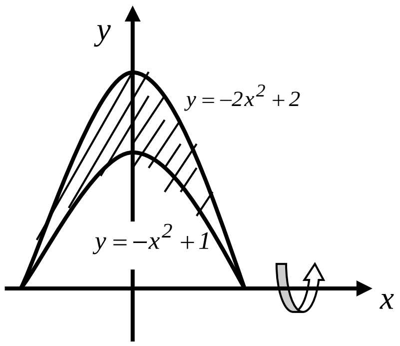
Volume By Revolution

State the integral formula represent the revolved volume arise by revolve the given shadow area :









Draw the area bounded by the given graphs, and find the arises revolved volume about the given axis:

(1) $x=1$; $y=1/x$; $y=0$; $x=3$ around x-axis .

(2) $x=-2$; $y=x^3$; $y=0$ around x-axis .

(3) $2y=x$; $y^2=x$; around y-axis

(4) $x-y=-1$; $x+y=1$; $x=2$ around y-axis .

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REFERENCES

1. R. Garg, MATHEMATICS - I (Calculus and Linear Algebra), KHANNA BOOK PUBLISHING CO. PVT. LTD. (2021).

2. E. Mendelson, Schaum's Outline of Calculus-McGraw Hill (2021).

3. R. L. Garg, Nishu Gupta, Engineering Mathematics Volume I-Pearson Education (2015).

4. E. Mendelson, Schaum's outline of beginning calculus-McGraw-Hill (2008).

5. K. Binmore and J. Davies, Calculus: Concepts and Methods, Cambridge University Press (2001).

6. Fred Safier, Schaum's Outline of Precalculus-McGraw-Hill (1997).

7. E. W. Swokowski - Calculus the classic edition (1991).

8. F. Ayres and E. Mendelson - Schaum's outline of theory and problems of differential and integral calculus-McGraw-Hill (1990).