Control Course

REFERENCE BOOK: Discrete Time Control Systems BY katsuhiko ogata

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CHAPTER(1)

Introduction To Discrete-time Control Systems

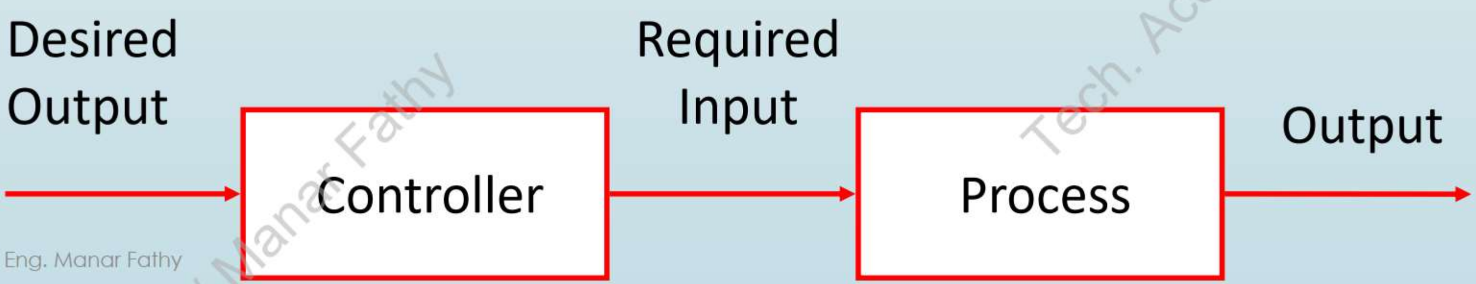
What is a System?

System: Block box that takes input signal(s) and converts to output signal(s).

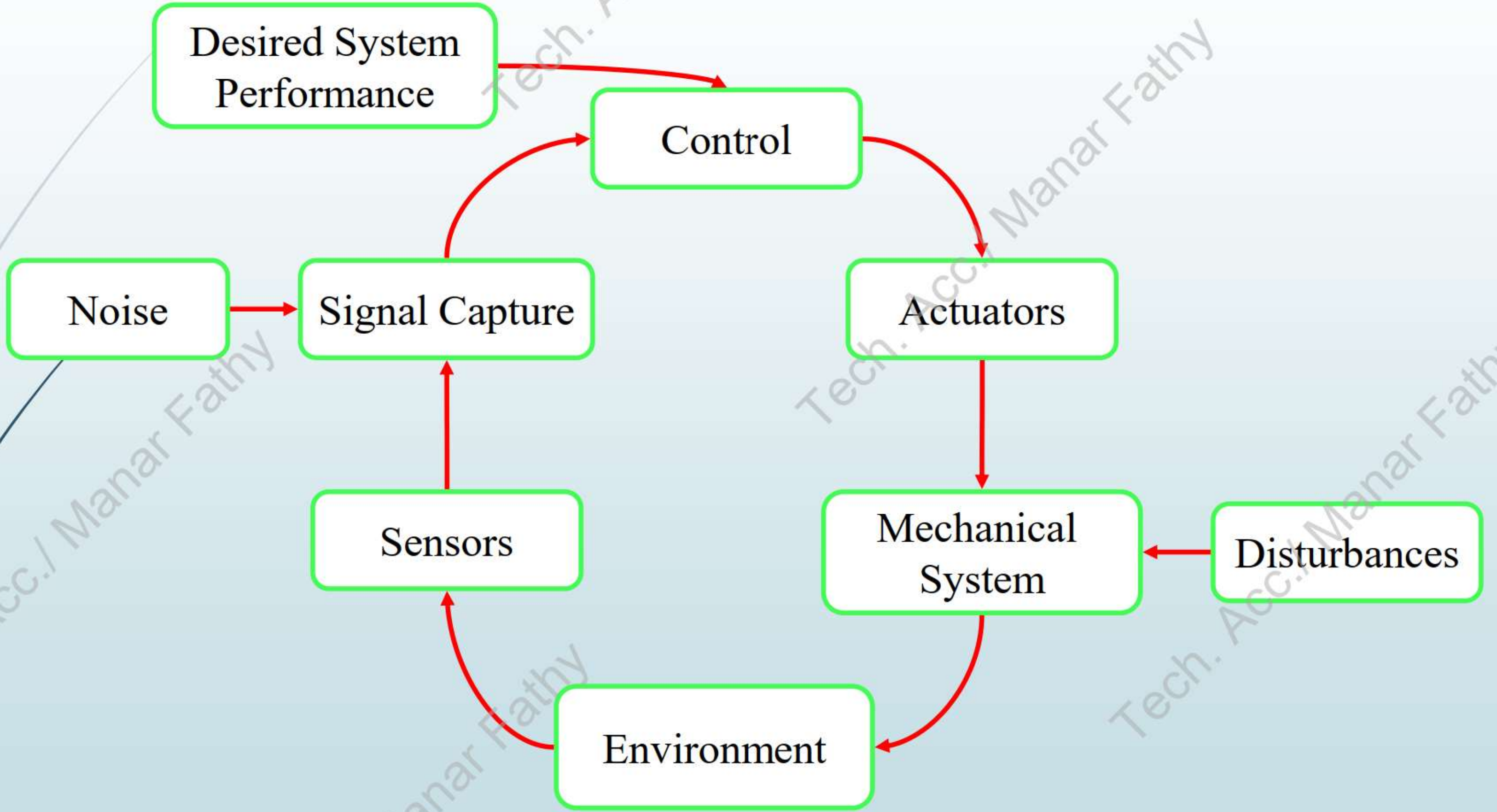
- **Continuous-Time System:**

What is a Control System

- **A Process that needs to be controlled:**
  - To achieve a desired output
  - By regulating inputs
- **A Controller: a mechanism, circuit or algorithm**
  - Provides required input
  - For a desired output

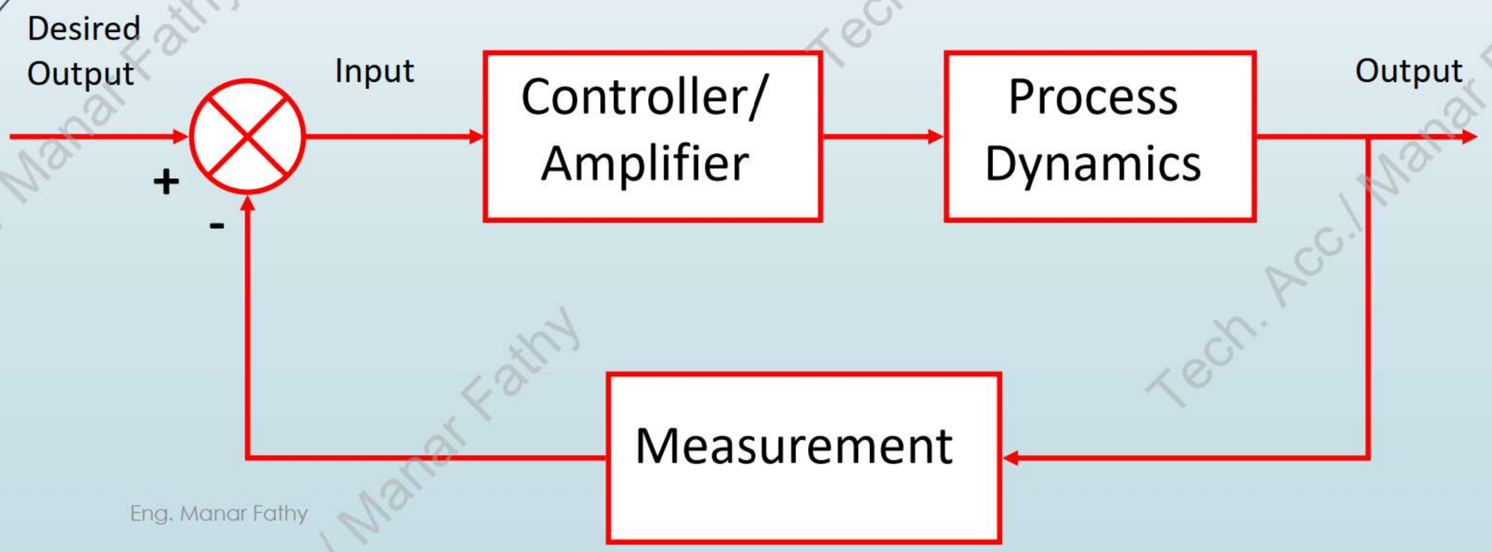


Model of Control System

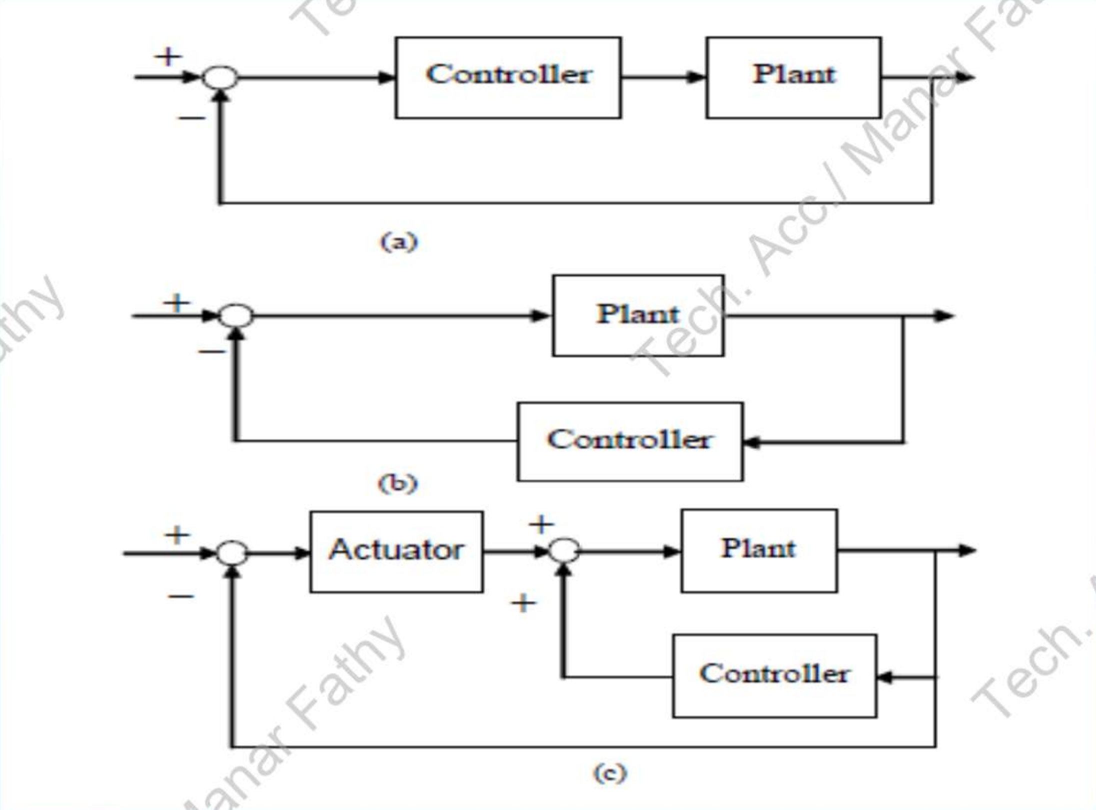


Closed Loop Control System

- **<u>Closed-loop</u> control takes account of actual output and compares this to desired output**



Types of Closed Loop Controller



Introduction Digital Control System

- A rapid increase in the use of digital controllers in control systems.
  - Intelligent motion in industrial robots
  - Optimization of fuel economy in automobiles
  - Refinements in the operation of appliances and machines
  - *Decision-making capability and flexibility in the control program – major advantages.*

- The current trend toward digital rather than analog:
  - Due to the availability of low-cost digital computers and the advantages found in working with digital signals.

Introduction

- **Systems types**
  - A *linear* system
    - one which the principle of superposition applies
    - may be described by linear differential or linear difference equations
  - A *time-invariant*
    - The coefficients in the differential equation or difference equation do not vary with time (i.e., one in which the properties of the system do not change with time)
- **Discrete-time control systems and continuous-time control systems**
  - Continuous-time control systems:
    - Signals are continuous in time
    - Described by differential equations
  - Discrete-time control systems:
    - One or more variables can change only at discrete instances of time
    - The interval between two discrete instants is taken to be sufficiently short.

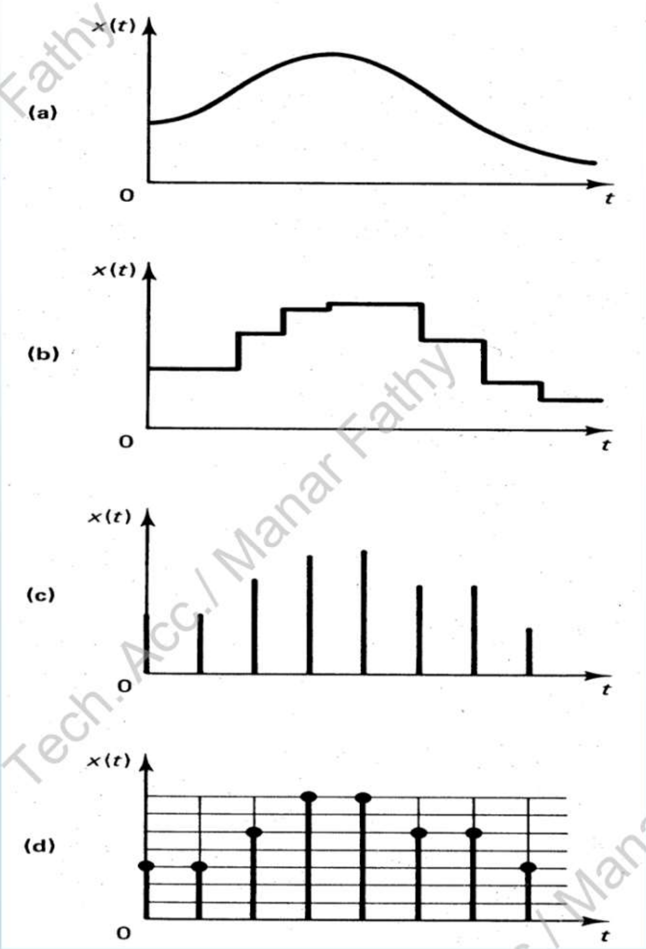
Mathematical comparison between analog and digital time control systems

System	Analytical model		
	Time domain	Frequency domain	
Continuous-time systems	Differential equations or state-space equations	Laplace transfer function (s-transfer function)	s-plane analysis and design techniques (Routh-Hurwitz stability criterion, root locus techniques, Bode plots, etc.)
Discrete-time systems	Difference equations or discrete-time state-space equations	Impulse transform function (z-transfer function)	z-plane analysis and design techniques (Jury stability test, modified root locus techniques, etc.)

Introduction

- Types of signals
  - Quantization: the process of representing a variable by a set of distinct values.
  - A discrete-time signal is a signal defined only at discrete instants of time.
  - A digital signal is a discrete-time signal with quantized amplitude and can be represented by a sequence of numbers.

  - *The use of digital controller requires quantization of signals both in amplitude and in time.*



Introduction

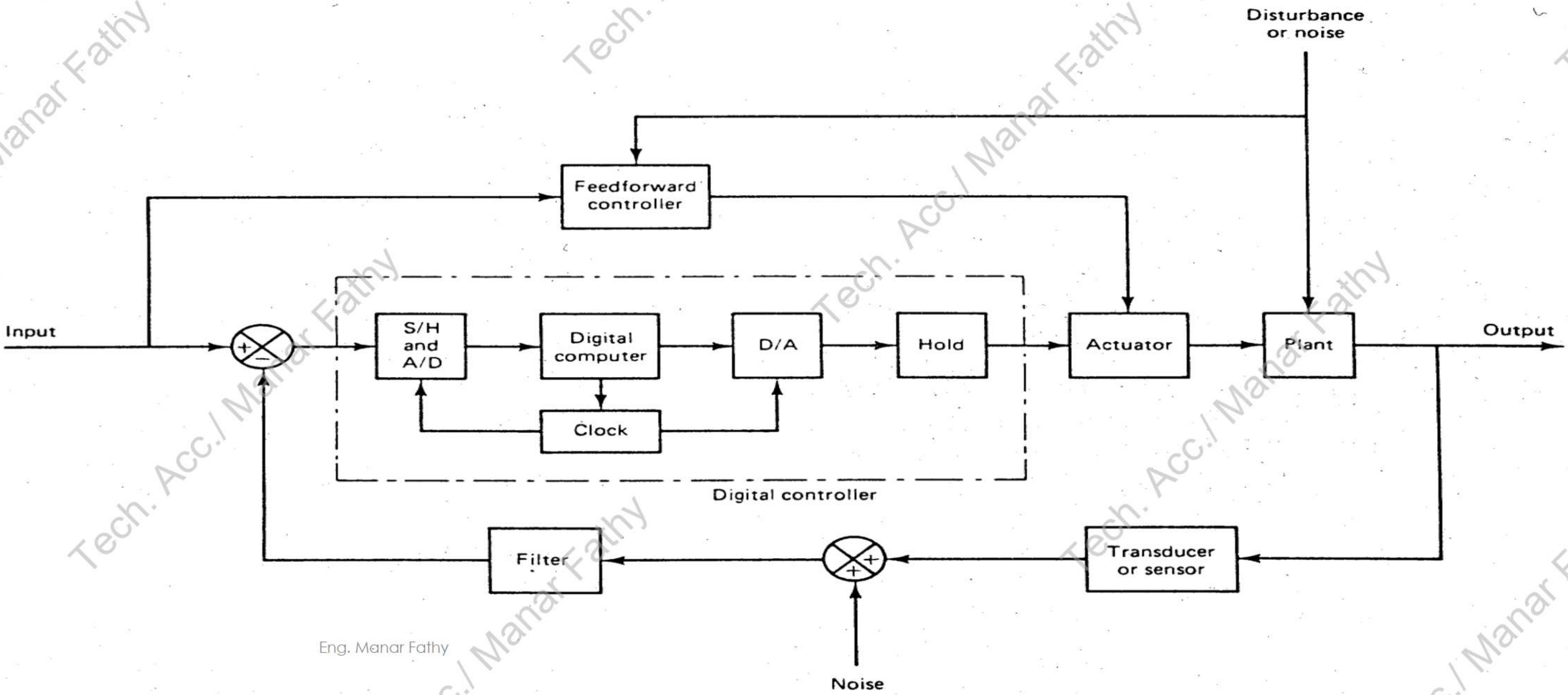
- Sampling and Quantization

Introduction

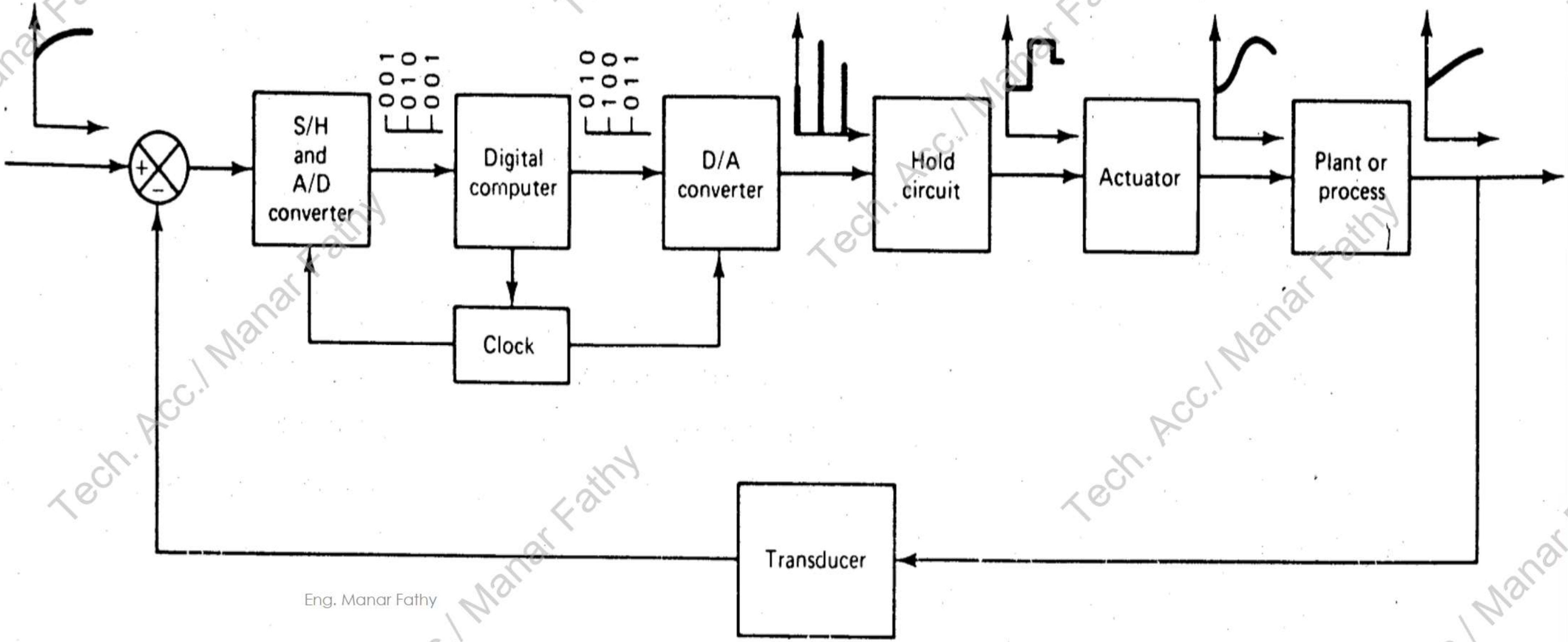
- **Sampling processes**
  - The sampling of a continuous-time signal replaces the original continuous-time signal by a sequence of values at discrete time points.
  - The sampling process is usually followed by a quantization process.
  - It is important to note that occasionally the sampling operation is entirely fictitious and has been introduced to simplify the analysis of control systems.

  - Many digital control systems are based on continuous-time design techniques.
    - A thorough knowledge of them is highly valuable in designing discrete-time control systems.

Digital Control Systems



Digital Control Systems



Digital Control Systems

- **Sample-and-Hold (S/H)**
  - A circuit that receives an analog input signal and holds this signal at a constant value for a specified period.
- **Analog-to-Digital Converter (A/D)**
  - A device that converts an analog signal into a digital signal, usually a numerically coded signal.
  - Such a converter is need as an interface between an analog component and a digital component.
  - A S/H is often a integral part of a commercially available A/D converter.
  - The approximation process by the limited number of bits is called quantization.
- **Digital-to-Analog Converter (D/A)**
  - A device that converts a digital signal into an analog signal.
  - A converter is needed as an interface between a digital component and an analog component.

Digital Control Systems

- **Plant or Process**
  - Any physical object to be controlled.
  - The most difficult part in the design of control systems may lie in the accurate modeling of a physical plant or process.
    - In designing a digital controller, it is necessary to recognize the fact that the mathematical model of a plant or process in many cases is only an approximation of the physical one.
- **Transducer**
  - A device that converts an input signal into an output signal of another form.
    - Ex: converting a pressure signal into a voltage output

Digital Control Systems

- **Types of sampling operations**
  - Periodic sampling
    - The sampling instants are equally spaced (t_k).
    - The most conventional type of sampling operation.
  - Multiple-order sampling
    - The pattern of the t_k's is repeated periodically.
  - Multiple-rate sampling
    - A digital control system may have different sampling periods in different feedback paths or may have multiple sampling rates.
      - To sample slowly in a loop involving a large time constant, while in a loop involving only small-time constants the sampling rate must be fast.
  - Random sampling
    - The sampling instants are random, or t_k is a random variable.

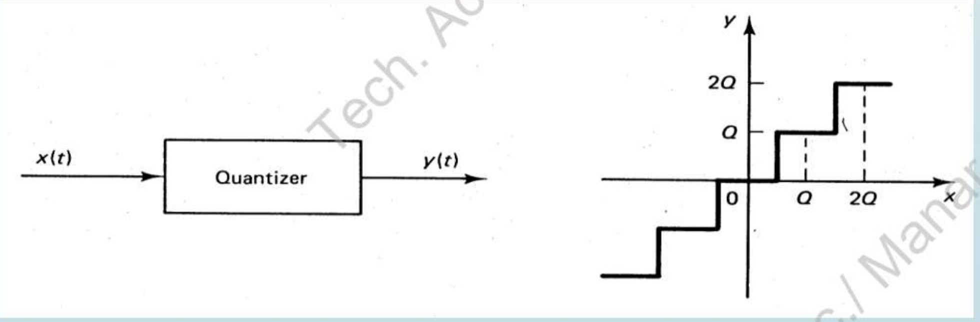
Quantization and Quanization Error

- **Quantizing**
  - The process of representing a continuous or analog signal by a finite number of discrete states.
  - The quantization level *Q* is defined as the range between two adjacent decision points:

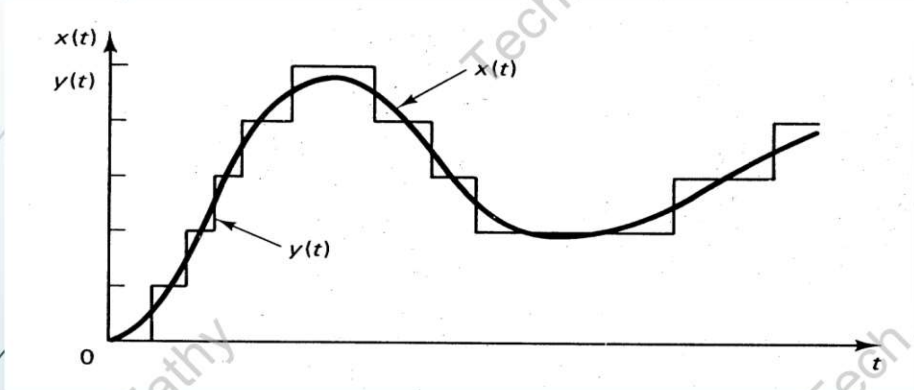
$$Q = \frac{FSR}{2^n}$$

FSR:full-scale range

- **Quantization Error**
  - A/D conversion results in a finite resolution.
  - Varies between 0 and ±½*Q*.



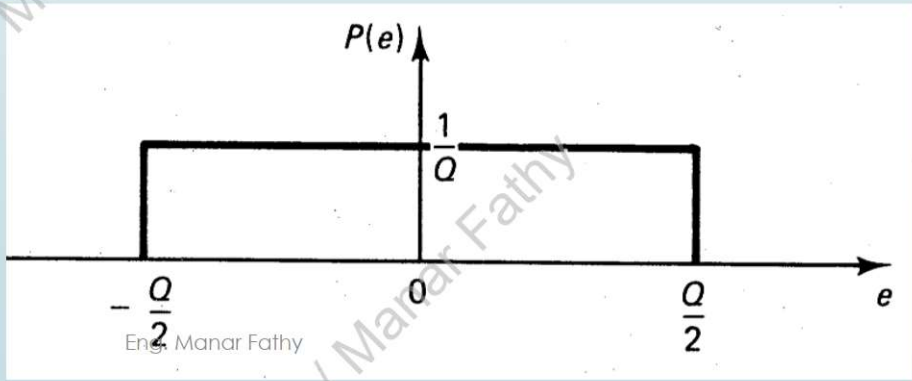
Quantization and Quanization Error



- Quantization Error (cont.)

$$e(t) = x(t) - y(t)$$

$$0 \le |e(t)| \le \tfrac{1}{2} Q$$

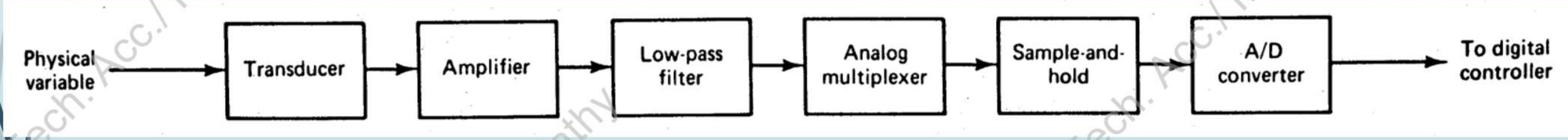


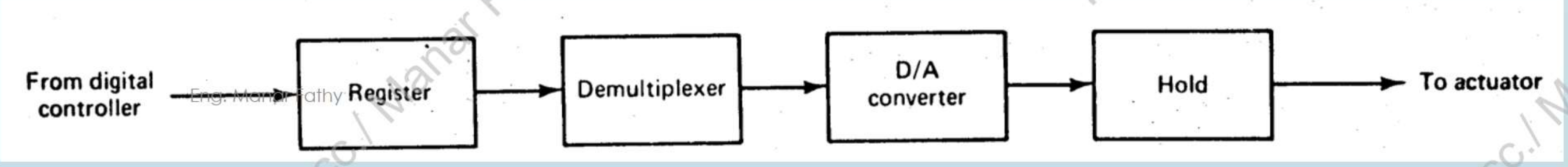
- For a small quantization level *Q*, the nature of the quantization error is like that of random noise.

$$\sigma^2 = E[e(t) - \bar{e}(t)]^2 = \frac{1}{Q}\int_{-Q/2}^{Q/2} \xi^2 d\xi = \frac{Q^2}{2}$$

Data Acquisition, Conversion, and Distribution Systems

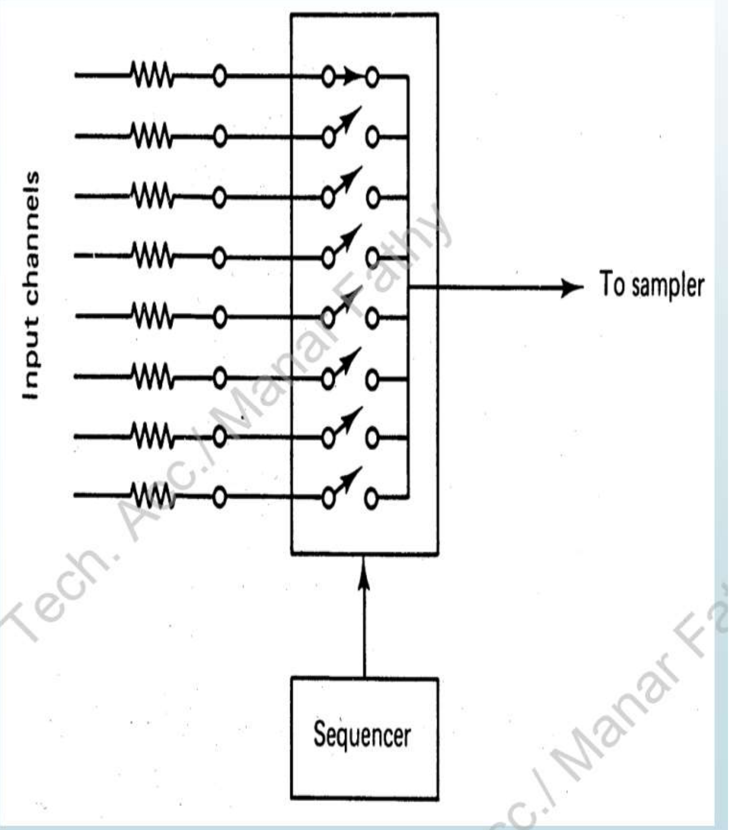
- <u>Operations involved in the signal conversion:</u>
  - Multiplexing and demultiplexing
  - Sample and hold
  - Analog-to-digital conversion (quantizing and encoding)
  - Digital-to-analog conversion (decoding)





Data Acquisition, Conversion, and Distribution Systems

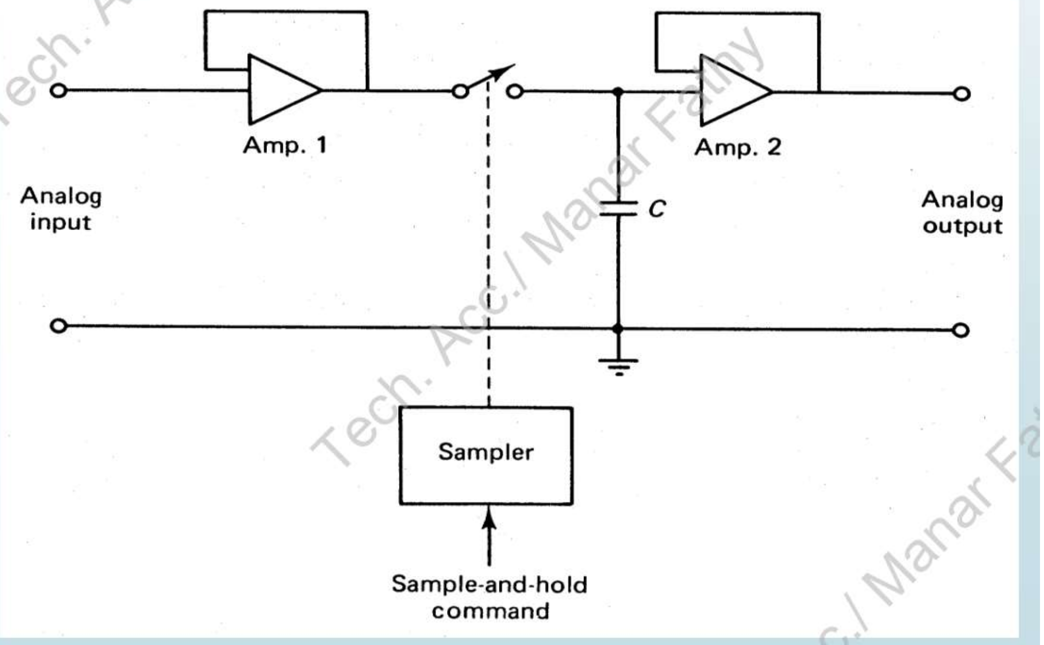
- Analog multiplexer
  - A device that performs the function of time-sharing an A/D converter among many analog channels.
  - At a given instant of time, only one switch is in the "on" position.
  - During the connection time the S/H samples the signal voltage and holds its value, while the ADC converts the analog value into digital data (binary numbers).



Data Acquisition, Conversion, and Distribution Systems

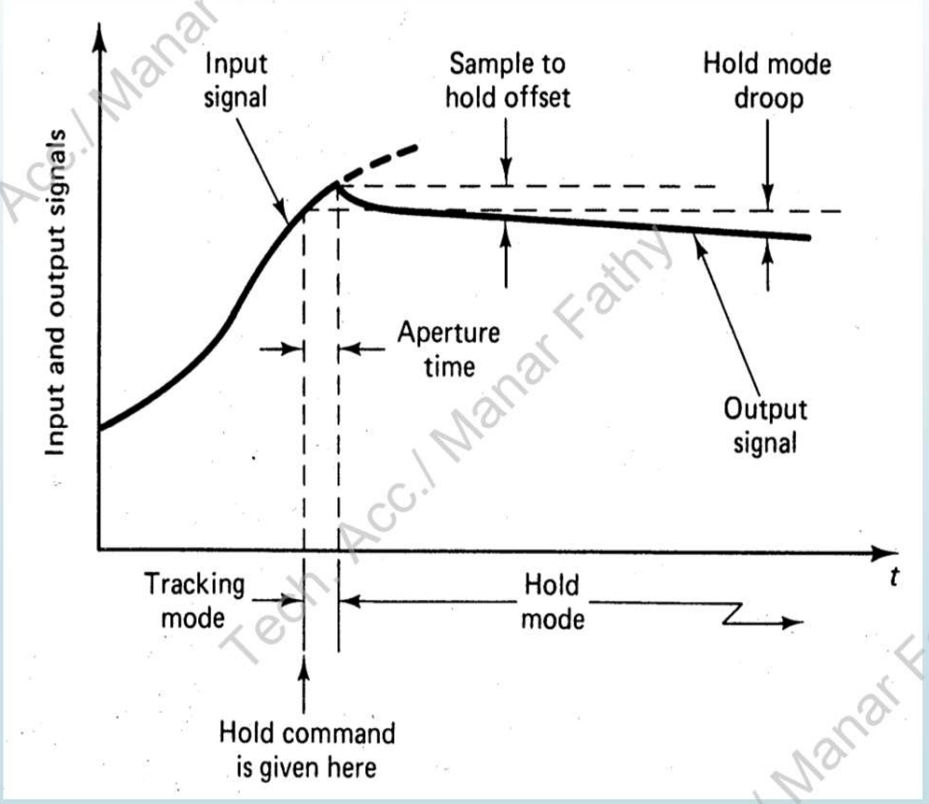
- Demultiplexer
  - A device separates the composite output digital data from the digital controller into the original channels.

- Sample-and-Hold Circuits
  - A sampler converts an analog signal into a train of amplitude-modulated pulses.
  - The hold circuit holds the value of the sampled pulse signal over a specified period.



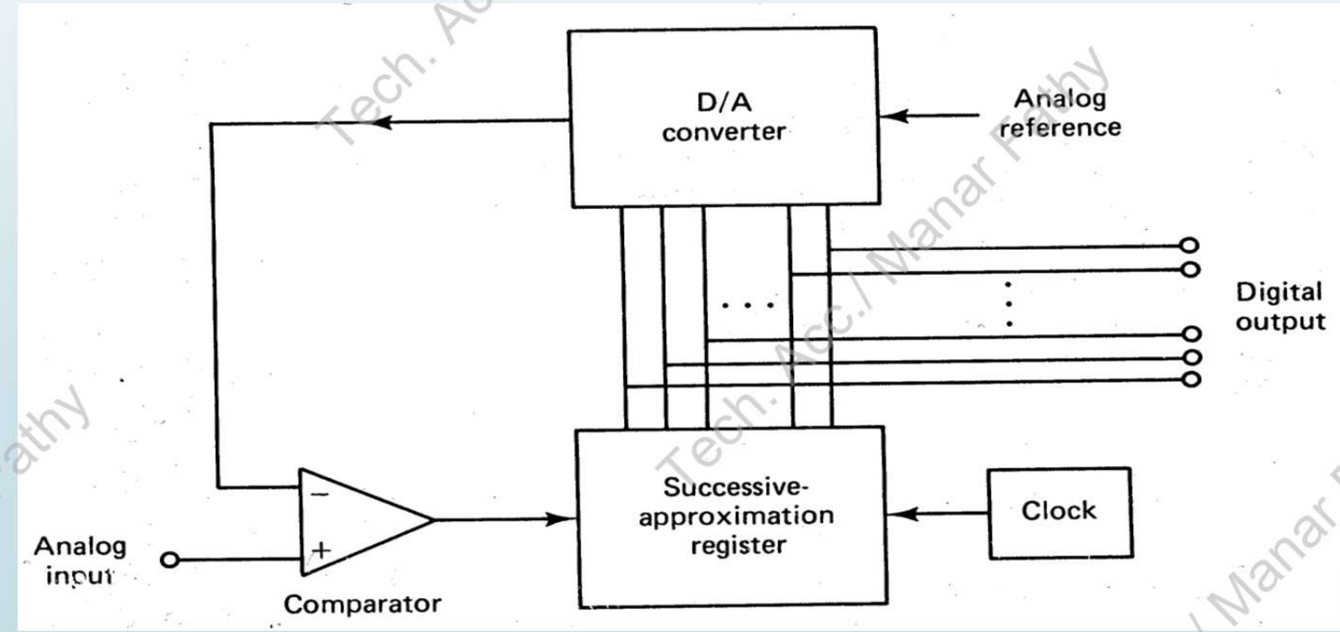
Data Acquisition, Conversion, and Distribution Systems

- Sample-and-Hold Circuits (cont.)
  - Two modes of operation:
    - The tracking mode – when the switch is closed, the charge on the capacitor in the circuit tracks the input voltage.
    - Hold mode – when the switch is open, the operating mode is the hold mode, and the capacitor voltages holds constant for a specified time of period.



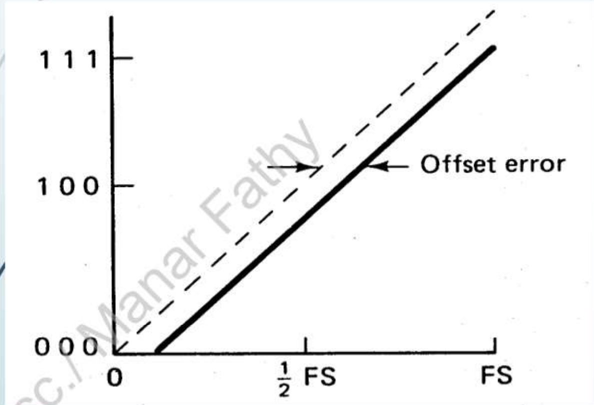
Data Acquisition, Conversion, and Distribution Systems

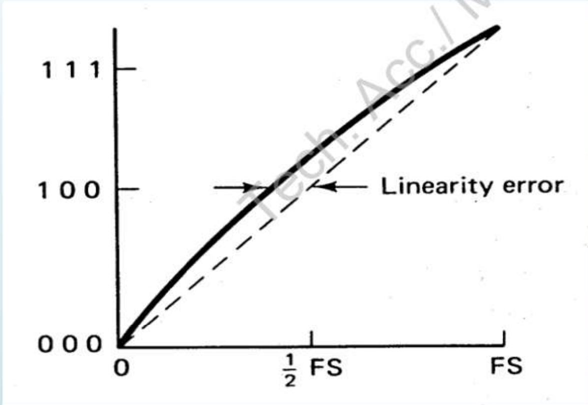
- Types of ADCs
  - Successive-approximation type
  - Integrating type
  - Counter type
  - Parallel type

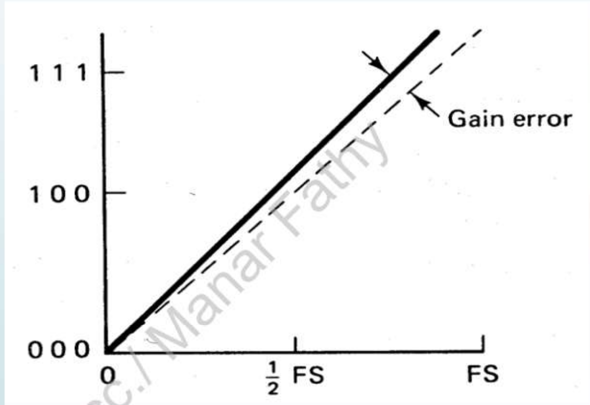


Data Acquisition, Conversion, and Distribution Systems

- Errors in ADCs

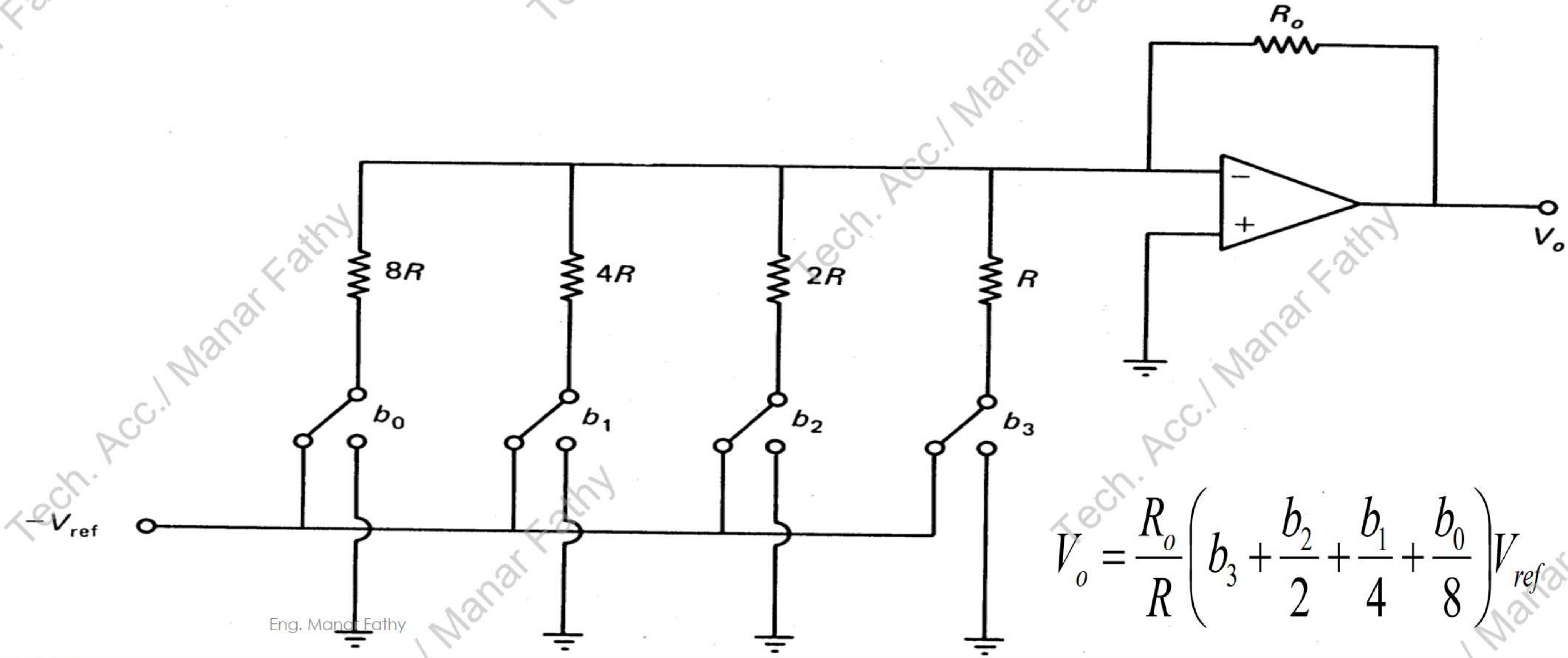






- Digital-to-Analog Converters
  - DAC using weighted registers
  - DAC using an R-2R ladder circuits

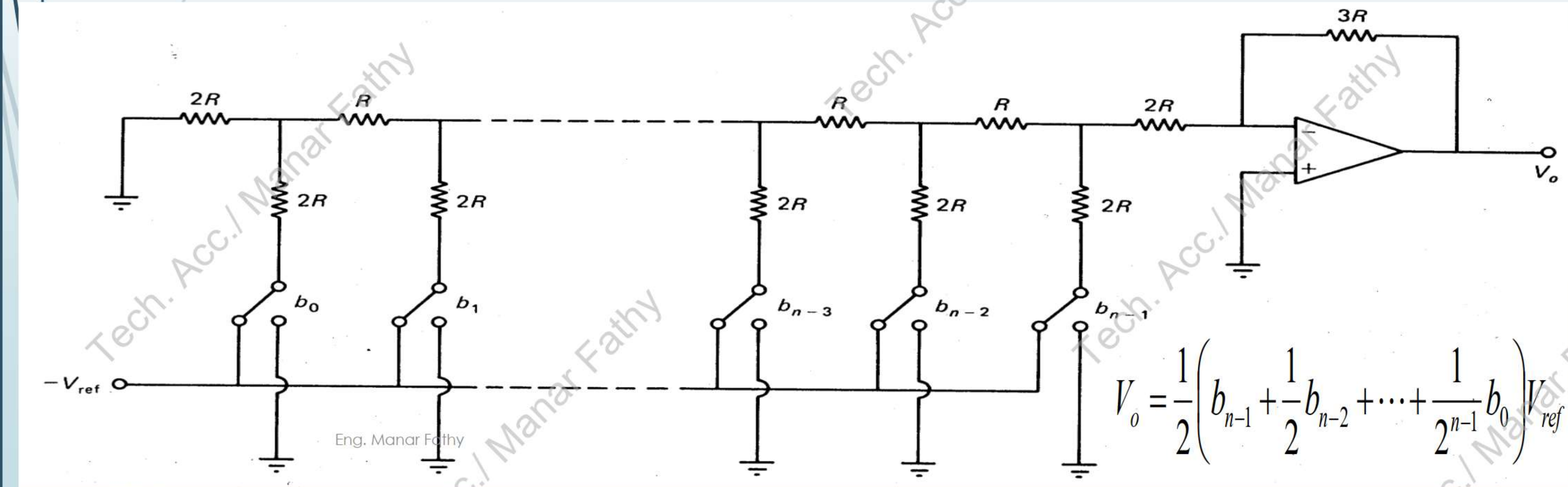
Data Acquisition, Conversion, and Distribution Systems



$$V_o = \frac{R_o}{R}\left(b_3 + \frac{b_2}{2} + \frac{b_1}{4} + \frac{b_0}{8}\right)V_{ref}$$

Data Acquisition, Conversion, and Distribution Systems

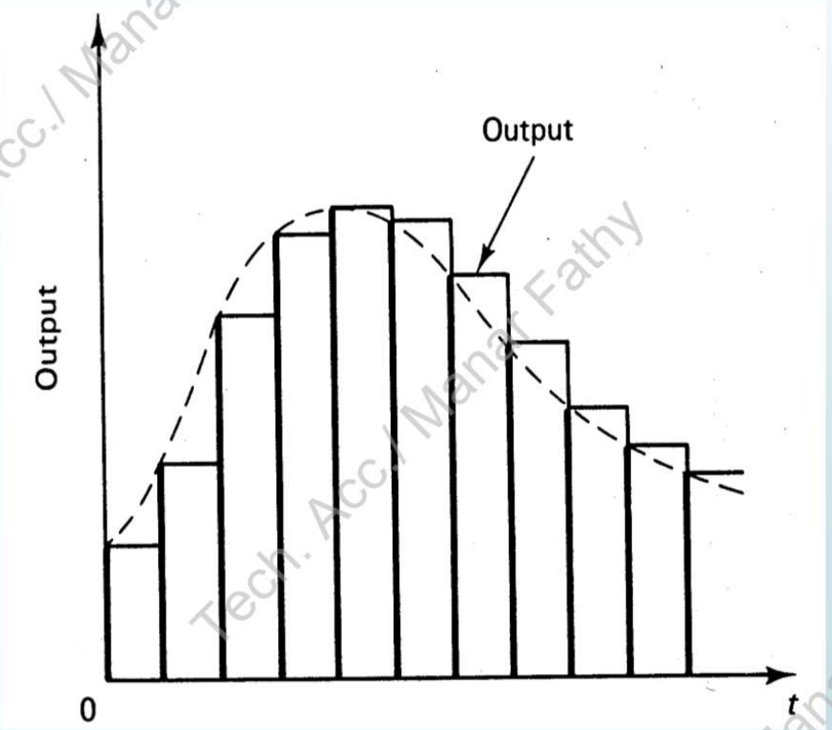
- Digital-to-Analog Converters (cont.)
  - DAC using an R-2R ladder circuits



$$V_o = \frac{1}{2}\left(b_{n-1} + \frac{1}{2}b_{n-2} + \cdots + \frac{1}{2^{n-1}}b_0\right)V_{ref}$$

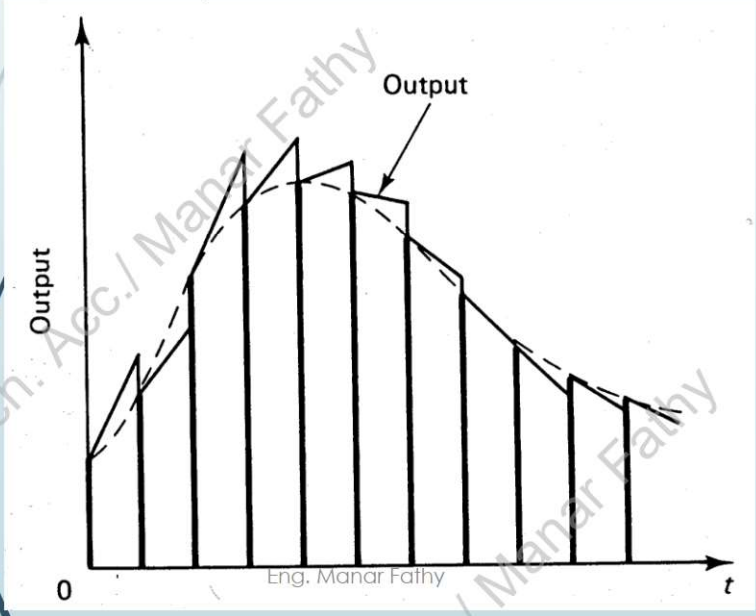
Data Acquisition, Conversion, and Distribution Systems

- Reconstructing the Input Signal by Hold Circuits
  - The purpose of the hold operation is to fill the spaces between sampling periods, and this roughly reconstruct the original analog input signal.
  - Zero-order hold

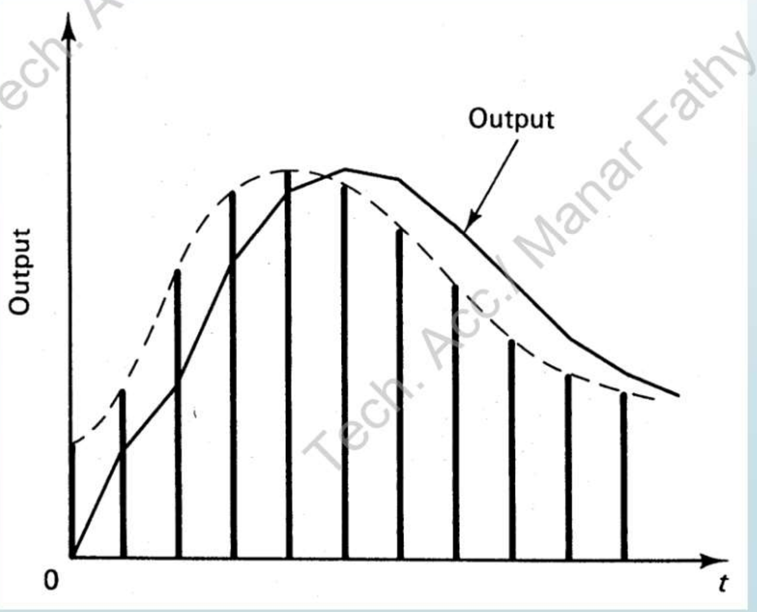


Data Acquisition, Conversion, and Distribution Systems

- Reconstructing the Input Signal by Hold Circuits (cont.)
  - First-order hold



  - Polygonal hold



Concluding Comments

- Digital controllers and Analog controllers
  - Digital controllers
    - operate only on numbers.
    - are extremely versatile.
    - Operations being performed can be changed by simply issuing a new program.
    - Digital components are rugged in construction, highly reliable, and often compact and lightweight.
- Digital control of processes
  - It is possible to consider all process variables, and thereby to accomplish optimal control of industrial processes.
  - Flexibility : ease of changing control schemes by reprogramming.

FELICITOUSLY

END OF CHAPTER (1)

CHAPTER(2)

The Z-Transform

Z - Transform Introduction

- The Z-transform plays the same role in the analysis of discrete-time signals and LTI systems as the Laplace transform does in the continuous-time signals and LTI systems.
- It offers the techniques for digital filter design and frequency analysis of digital signals.

Definition of z-transform:

The z-transform of the discrete-time $\boldsymbol{x}[\boldsymbol{n}]$ is given by:

$$x[n] \xrightarrow{\mathcal{Z}} X(z)$$

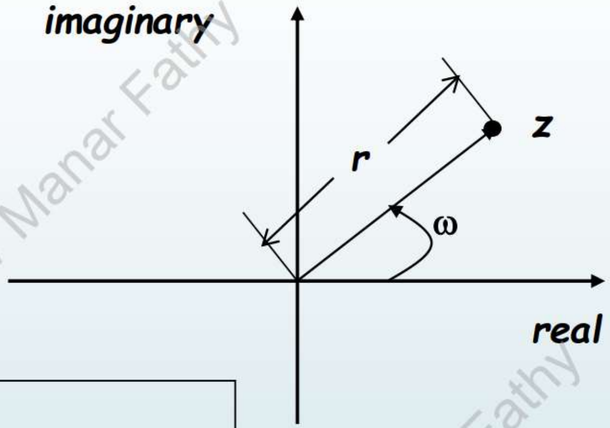
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

Where z is a complex variable

Z - Transform Introduction

Z is a complex variable:

$$z = re^{j\omega} = r\cos\omega + jr\sin\omega$$



$$z^n = r^n e^{j\omega n} = \underbrace{r^n \cos\omega n}_{\text{real part}} + j\underbrace{r^n \sin\omega n}_{\text{imaginary part}}$$

rate of growth is → *r*
frequency is → ω

Z - Transform Introduction

For a ***causal*** sequence:

$x(n) = 0 \; for \; n < 0$

$$X(z) = Z(x(n)) = \sum_{n=0}^{\infty} x(n) z^{-n}$$
$$= x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

Some simple pairs:

- finite-length sequence

$$\underset{\substack{\uparrow \\ n=0}}{[1} \quad 2 \quad -1 \quad 3] \xleftrightarrow{Z} 1 + 2z^{-1} - 1z^{-2} + 3z^{-3}$$

- impulse $\delta(n) \xleftrightarrow{Z} \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} = z^{-0} = 1$

Z-transform of the Unit impulse

Let x(kT) = δ(kT)

$$\delta(\mathrm{kT}) = \begin{cases} 1 & \text{for k} = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$X(z) = \sum_{k=0}^{\infty} x[kT]\, z^{-k} = \sum_{k=0}^{\infty} \delta(\mathrm{kT})\, z^{-k} = 1$$

Z-transform of the Shifted Unit impulse

Let x(kT) = δ(kT-qT)

$$\delta(\mathrm{kT - qT}) = \begin{cases} 1 & \text{for } \mathrm{k = q} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$X(z) = \sum_{k=0}^{\infty} x(kT)\, z^{-k} = \sum_{k=0}^{\infty} \delta(\mathrm{kT - qT})\, z^{-k} = z^{-q}$$

Z-transform of the Unit-Step Function

Let x(kT) = u(kT)

$$u(\mathrm{kT}) = \begin{cases} 1 & \text{for } \mathrm{k} \geq 0 \\ 0 & \text{for } \mathrm{k} \prec 0 \end{cases}$$

Then

$$X(z) = \sum_{k=0}^{\infty} x(kT)\, z^{-k} = \sum_{k=0}^{\infty} u(\mathrm{kT})\, z^{-k} = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

Z-transform of the Sample Exponential

Let x(kT) = a^k u(kT)

$$a^k u(\mathrm{kT}) = \begin{cases} a^k & \text{for k} \geq 0 \\ 0 & \text{for k} \prec 0 \end{cases}$$

Then

$$X(z) = \sum_{k=0}^{\infty} x(kT)\, z^{-k} = \sum_{k=0}^{\infty} a^k u(\mathrm{kT})\, z^{-k} = \sum_{k=0}^{\infty} a^k z^{-k}$$

$$X(z) = \sum_{k=0}^{\infty} (az^{-1})^k = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

Z-transform of the Sinusoids

Let x(kT) = (cos ΩkT) u(kT)

$$\cos(\Omega n)\,u(\mathrm{n}) = \begin{cases} \cos(\Omega kT) & \text{for } k \geq 0 \\ 0 & \text{for } k \prec 0 \end{cases}$$

$$\because \cos(\Omega kT) = \frac{e^{j\Omega kT} + e^{-j\Omega kT}}{2}$$

Then $Z\{\cos(\Omega kT)\mathrm{u(k)}\} = \frac{1}{2} Z\{e^{j\Omega kT} u(kT) + e^{-j\Omega kT} u(kT)\}$

$$Z\{\cos(\Omega\, kT)\mathrm{u}(kT)\} = \frac{1}{2}\left[\frac{z}{z - e^{j\Omega T}} + \frac{z}{z - e^{-j\Omega T}}\right]$$

$$\boxed{Z\{\cos(\Omega\, kT)\mathrm{u}(kT)\} = \frac{1 - z^{-1}\cos(\Omega T)}{1 - 2z^{-1}\cos(\Omega T) + z^{-2}} = \frac{z^2 - z\cos(\Omega T)}{z^2 - 2z\cos(\Omega T) + 1}}$$

Similarly it can be shown:

$$\boxed{Z\{\sin(\Omega\, kT)\mathrm{u}(kT)\} = \frac{z^{-1}\sin(\Omega T)}{1 - 2z^{-1}\cos(\Omega T) + z^{-2}} = \frac{z\sin(\Omega T)}{z^2 - 2z\cos(\Omega T) + 1}}$$

Z-Transform Table

	Signal, $x(kT)$	Z-Transform, $X(z)$	Z-Transform, $X(z)$	ROC
1	$\delta(kT)$	1	1	all Z
2	$\delta(kT-q)$	z^{-q}	z^{-q}	$z \neq 0$
3	$u(kT)$	$\frac{1}{1-z^{-1}}$	$\frac{z}{z-1}$	$\|z\| > 1$
4	$a^k\, u(kT)$	$\frac{1}{1-az^{-1}}$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
5	$k\, u(kT)$	$\frac{1}{(1-z^{-1})^2}$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
6	$k\, a^k\, u(kT)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$
7	$\cos(\omega_0 kT)$	$\frac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\frac{z^2-z\cos(\omega_0)}{z^2-2z\cos(\omega_0)+1}$	$\|z\| > 1$
8	$\sin(\omega_0 kT)$	$\frac{z^{-1}\sin(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\frac{z\sin(\omega_0)}{z^2-2z\cos(\omega_0)+1}$	$\|z\| > 1$
9	$r^k \cos(\omega_0 kT)$	$\frac{1-r\,z^{-1}\cos(\omega_0)}{1-2r\,z^{-1}\cos(\omega_0)+r^2 z^{-2}}$	$\frac{z^2-r\,z\cos(\omega_0)}{z^2-2r\,z\cos(\omega_0)+r^2}$	$\|z\| > \|r\|$
10	$r^k \sin(\omega_0 kT)$	$\frac{r\,z^{-1}\sin(\omega_0)}{1-2r\,z^{-1}\cos(\omega_0)+r^2 z^{-2}}$	$\frac{r\,z\sin(\omega_0)}{z^2-2r\,z\cos(\omega_0)+r^2}$	$\|z\| > \|r\|$

Z-Transform Properties (1)

Linearity: $$Z(ax_1(n)+bx_2(n)) = aZ(x_1(n))+bZ(x_2(n))$$

$x_1(n)$ and $x_2(n)$ denote the sampled sequences, ***a*** and ***b*** are the arbitrary constants.

Example

Problem:

Find the z-transform of $$x(n) = u(n) - (0.5)^n u(n)$$

Solution:

Applying the linearity of the z-transform $$X(z) = Z(x(n)) = Z(u(n)) - Z(0.5^n(n))$$

Using z-transform Table

$$Z(u(n)) = \frac{z}{z-1}$$

$$Z(0.5^n u(n)) = \frac{z}{z-0.5}$$

Therefore, we get, $$X(z) = \frac{z}{z-1} - \frac{z}{z-0.5}$$

Z-Transform Properties (2)

$x(n) \xrightarrow{\text{Z-Transform}} X(z)$

Shift Theorem: $Z(x(n-m)) = z^{-m}X(z)$

Verification:

$$Z(x(n-m)) = \sum_{n=0}^{\infty} x(n-m)z^{-n}$$
$$= x(-m)z^{-0} + \cdots + x(-1)z^{-(m-1)} + x(0)z^{-m} + x(1)z^{-m-1} + \ldots$$

Since $x(n)$ is assumed to be causal: $x(-m) = x(-m+1) = \cdots = x(-1) = 0$

Then we achieve, $Z(x(n-m)) = x(0)z^{-m} + x(1)z^{-m-1} + x(2)z^{-m-2} + \ldots$

Factoring z^{-m} from Equation we get,

$$Z(x(n-m)) = z^{-m}\left(x(0) + x(1)z^{-1} + x(2)z^{-2} + \ldots\right) = z^{-m}X(z)$$

Z-Transform Properties (3)

Multiplication by n:

$$nx[n] \quad \Leftrightarrow \quad -z\frac{dX(z)}{dz}$$

Verification:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$\frac{dX(z)}{dz} = -n\sum_{n=-\infty}^{\infty} x[n]z^{-n-1} \quad \Rightarrow \quad -z\frac{dX(z)}{dz} = n\sum_{n=-\infty}^{\infty} x[n]z^{-n} = Z\{nx[n]\}$$

Z-Transform Properties (4)

Convolution:

In time domain, $$x(n) = x_1(n) * x_2(n) = \sum_{k=0}^{\infty} x_1(n-k)x_2(k)$$ eq.(1)

In Z-transform domain, $$X(z) = X_1(z)X_2(z)$$

Here, $X(z) = Z(x(n))$, $X_1(z) = Z(x_1(n))$, and $X_2(z) = Z(x_2(n))$.

Verification:

Taking the z-transform of eq.(1) $$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = \sum_{n=0}^{\infty} \underbrace{\sum_{k=0}^{\infty} x_1(n-k)x_2(k)}_{x(n)\ from\ eq.(1)} z^{-n}$$

$z^{-n} = z^{-k}z^{-(n-k)}$

$$X(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} x_2(k)z^{-k}x_1(n-k)z^{-(n-k)}$$

$$X(z) = \sum_{k=0}^{\infty} x_2(k)z^{-k} \sum_{n=0}^{\infty} x_1(n-k)z^{-(n-k)}$$

let $m = n - k$:

$$X(z) = \sum_{k=0}^{\infty} x_2(k)z^{-k} \sum_{m=0}^{\infty} x_1(m)z^{-m}$$

$$X(z) = X_2(z)X_1(z) = X_1(z)X_2(z)$$

Z-Transform Properties (5)

Initial and Final Value Theorems:

Initial Value Theorem:

- The value of x(n) as k → 0 is given by:

$$x(0) = \lim_{k \to 0} x(kT) = \lim_{z \to \infty} X(z)$$

Final Value Theorem:

- The value of x(n) as k → ∞ is given by:

$$x(\infty) = \lim_{k \to \infty} x(kT) = \lim_{z \to 1} [(z-1)X(z)]$$

Transfer Functions

- In addition to our normal transfer function components, such as summation and multiplication, we use one important additional component: delay (D).

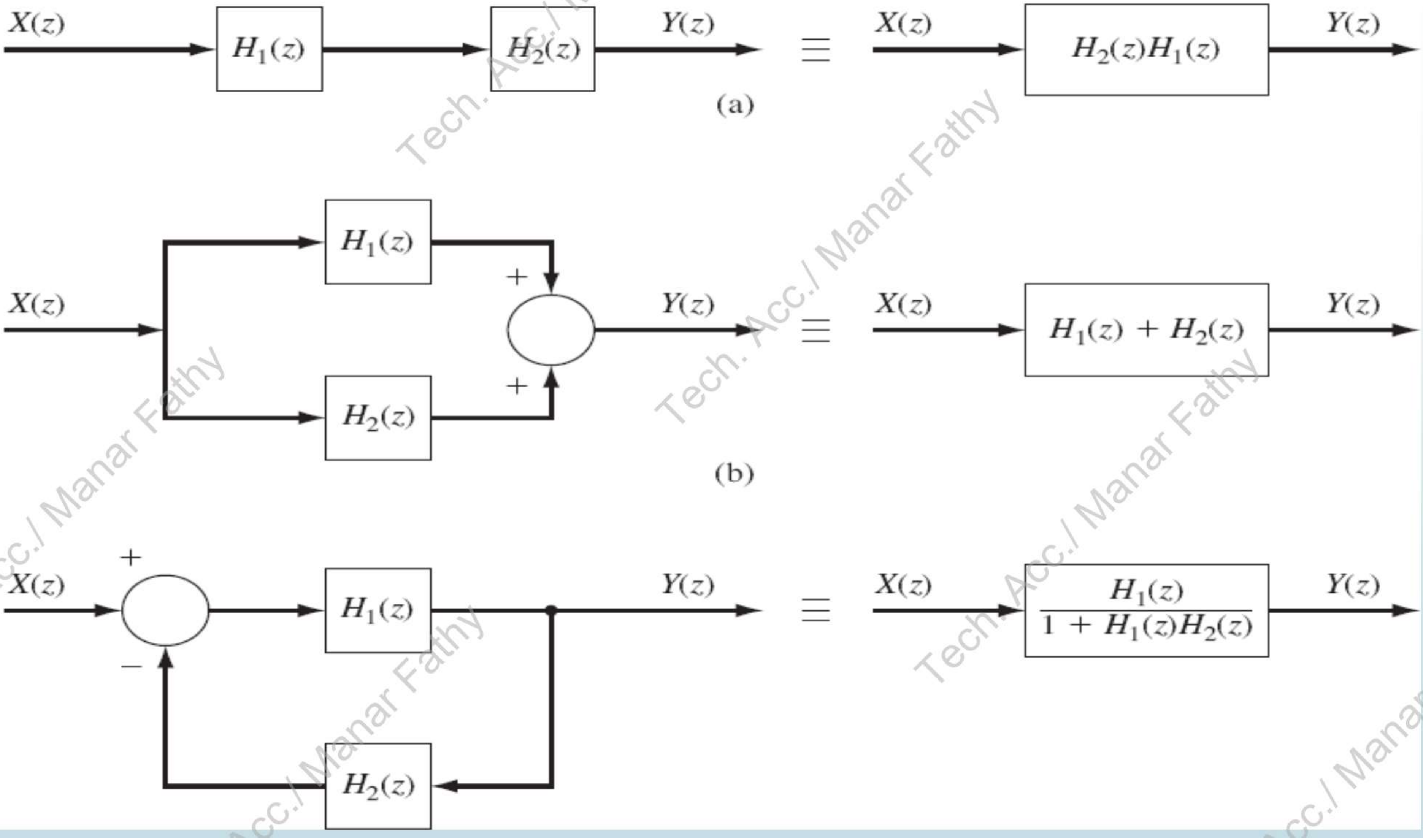
$$x[n] \longrightarrow \boxed{D} \longrightarrow y[n] = x[n-1]$$

$$x[n] \longrightarrow \boxed{z^{-1}} \longrightarrow y[n] = x[n-1]$$

$$Y(z) = z^{-1} X(z)$$

- This is often denoted by its z-transform equivalent.
- We can illustrate this with an example (assume initial conditions are zero):

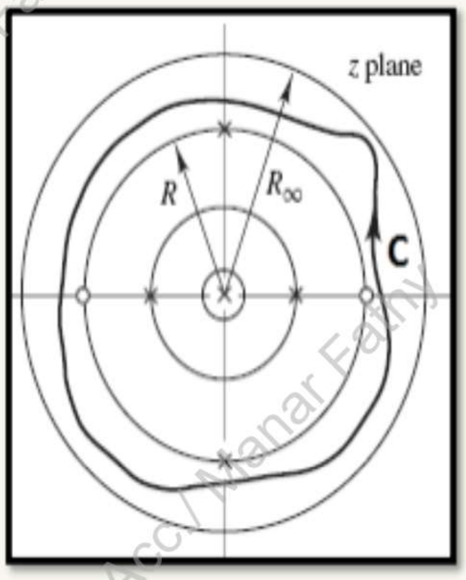
Basic Interconnections of Transfer Functions



Inverse z-Transform

Formal inverse z-transform is based on a Cauchy integral

$$x(kT) = Z^{-1}[X(z)] = \frac{1}{2\pi j}\oint_C X(z) z^{k-1} dz$$



Less formal ways sufficient most of the time

1) Direct or Long Division Method
2) Partial fraction expansion and Look-up Table
3) Inversion Integral Method (Residue-theorem)

Inverse Z-Transform: Power Series Expansion

Using Long Division to expand X(z) as a series

$$X(z) = x(0) + x(T)z^{-1} + x(2T)z^{-2} + \ldots = \sum_{k=0}^{\infty} x(kT)z^{-k}$$

Write the inverse transform as the sequence

$$x(kT) = \{x(0), x(T), x(2T), \ldots\}$$

Inverse z-Transform: Using Partial Fraction

Table Partial Fraction(s) and Formulas for Constant(s)

Partial fraction with the first-order real pole: $\frac{R}{z-p}$	$R = (z-p)\frac{X(z)}{z}\Big\vert_{z=p}$
Partial fraction with the first-order complex poles: $\frac{Az}{(z-P)} + \frac{A^* z}{(z-P^*)}$ $P^* =$ complex conjugate of P $A^* =$ complex conjugate of A	$A = (z-P)\frac{X(z)}{z}\Big\vert_{z=P}$
Partial fraction with mth-order real poles: $\frac{R_m}{(z-p)} + \frac{R_{m-1}}{(z-p)^2} + \cdots + \frac{R_1}{(z-p)^m}$	$R_k = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left((z-p)^m \frac{X(z)}{z}\right)\Big\vert_{z=p}$

k from m to 1

Z-transform method for Solution of Difference Equations

Transfer Function Representation

First - Order Case

- Let $y(kT) + ay(kT - T) = bx(kT)$
- Then take the z-transform to get:
  - $Y(z) + a z^{-1} Y(z) = b X(z)$
- Simplifying:
  - $Y(z)(1 + az^{-1}) = b X(z)$
  - $Y(z) = (b X(z))/(1 + az^{-1})$
- And we have the transfer function H(z):
  - $Y(z) = H(z) X(z)$
  - so ➔ $H(z) = \frac{b z}{z + a}$
- By inverse z-transform: $y(kT) = b(-a)^k u(kT)$

FELICITOUSLY

END OF CHAPTER (2)