Z-Transform Table

	Signal, $x(kT)$	Z-Transform, $X(z)$	Z-Transform, $X(z)$	ROC
1	$\delta(kT)$	1	1	all Z
2	$\delta(kT-q)$	z^{-q}	z^{-q}	$z \neq 0$
3	$u(kT)$	$\frac{1}{1-z^{-1}}$	$\frac{z}{z-1}$	$\|z\| > 1$
4	$a^k\, u(kT)$	$\frac{1}{1-az^{-1}}$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
5	$k\, u(kT)$	$\frac{1}{(1-z^{-1})^2}$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
6	$k\, a^k\, u(kT)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$
7	$\cos(\omega_0 kT)$	$\frac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\frac{z^2-z\cos(\omega_0)}{z^2-2z\cos(\omega_0)+1}$	$\|z\| > 1$
8	$\sin(\omega_0 kT)$	$\frac{z^{-1}\sin(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\frac{z\sin(\omega_0)}{z^2-2z\cos(\omega_0)+1}$	$\|z\| > 1$
9	$r^k\cos(\omega_0 kT)$	$\frac{1-r\,z^{-1}\cos(\omega_0)}{1-2r\,z^{-1}\cos(\omega_0)+r^2 z^{-2}}$	$\frac{z^2-r\,z\cos(\omega_0)}{z^2-2r\,z\cos(\omega_0)+r^2}$	$\|z\| > \|r\|$
10	$r^k\sin(\omega_0 kT)$	$\frac{r\,z^{-1}\sin(\omega_0)}{1-2r\,z^{-1}\cos(\omega_0)+r^2 z^{-2}}$	$\frac{r\,z\sin(\omega_0)}{z^2-2r\,z\cos(\omega_0)+r^2}$	$\|z\| > \|r\|$

Eng. Manar Fathy

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$

Tech. Acc./ Manar Fathy

Example 3

Problem:

Find the z-transform for each of the following sequences:

a. $x(n) = 10\sin(0.25\pi n)u(n)$ **b.** $x(n) = e^{-0.1n}\cos(0.25\pi n)u(n)$

Solution:

a. From line 9 in the Table: $X(z) = 10Z(\sin(0.2\pi n)u(n))$

$$= \frac{10\sin(0.25\pi)z}{z^2 - 2z\cos(0.25\pi) + 1} = \frac{7.07z}{z^2 - 1.414z + 1}$$

b. From line 14 in the Table: $X(z) = Z(e^{-0.1n}\cos(0.25\pi n)u(n)) = \frac{z(z - e^{-0.1}\cos(0.25\pi))}{z^2 - 2e^{-0.1}\cos(0.25\pi)z + e^{-0.2}}$

$$= \frac{z(z - 0.6397)}{z^2 - 1.2794z + 0.8187}$$

Tech. Acc./ Manar Fathy

Example 5

Problem:

Determine the z-transform of $y(n) = (0.5)^{(n-5)} \cdot u(n-5)$

where $u(n-5) = 1$ for $n \geq 5$ and $u(n-5) = 0$ for $n<5$.

Solution:

Using shift theorem,

$$Y(z) = Z\left[(0.5)^{n-5}u(n-5)\right] = z^{-5}Z\left[(0.5)^{n}u(n)\right]$$

Using z-transform table, line 6:

$$Y(z) = z^{-5} \cdot \frac{z}{z-0.5} = \frac{z^{-4}}{z-0.5}$$

Tech. Acc./ Manar Fathy

Example 6

Problem:

Given the sequences, $x_1(n) = 3\delta(n) + 2\delta(n-1)$

$x_2(n) = 2\delta(n) - \delta(n-1)$

Find the z-transform of the convolution.

Solution:

Applying the z-transform of the two sequences,

$$Z[x_1(n)] \Rightarrow X_1(z) = 3 + 2z^{-1}$$

$$Z[x_2(n)] \Rightarrow X_2(z) = 2 - z^{-1}$$

Therefore we get,

$$X(z) = X_1(z)X_2(z) = (3 + 2z^{-1})(2 - z^{-1})$$
$$= 6 + z^{-1} - 2z^{-2}$$

Tech. Acc./ Manar Fathy

Inverse z-Transform: Examples

The inverse z-transform for the function $X(z)$ is defined as: $x(n) = Z^{-1}(X(z))$

Example 7 Find the inverse z-transform of $X(z) = 2 + \frac{4z}{z-1} - \frac{z}{z-0.5}$

Solution We get, $x(n) = 2Z^{-1}(1) + 4Z^{-1}\left(\frac{z}{z-1}\right) - Z^{-1}\left(\frac{z}{z-0.5}\right)$

Using table, $x(n) = 2\delta(n) + 4u(n) - (0.5)^n u(n)$

Example 8 Find the inverse z-transform of $X(z) = \frac{5z}{(z-1)^2} - \frac{2z}{(z-0.5)^2}$

Solution We get, $x(n) = Z^{-1}\left(\frac{5z}{(z-1)^2}\right) - Z^{-1}\left(\frac{2z}{(z-0.5)^2}\right) = 5Z^{-1}\left(\frac{z}{(z-1)^2}\right) - \frac{2}{0.5}Z^{-1}\left(\frac{0.5z}{(z-0.5)^2}\right)$

Using table, $x(n) = 5nu(n) - 4n(0.5)^n u(n)$

Tech. Acc./ Manar Fathy

Inverse z-Transform: Examples

Example 9 Find the inverse z-transform of $X(z) = \frac{10z}{z^2 - z + 1}$

Solution Since, $X(z) = \frac{10z}{z^2 - z + 1} = \left(\frac{10}{\sin(a)}\right)\frac{\sin(a)z}{z^2 - 2z\cos(a) + 1}$ From line 9 in the Table

By coefficient matching, $-2\cos(a) = -1$

Hence, $\cos(a) = 0.5$, and $a = 60^0$. ⇨ $\sin(a) = \sin(60^0) = 0.866$

Therefore, $x(n) = \frac{10}{\sin(a)} Z^{-1}\left(\frac{\sin(a)z}{z^2 - 2z\cos(a) + 1}\right) = \frac{10}{0.866}\sin(n\cdot 60^0) = 11.547\sin(n\cdot 60^0)$

Example 10 Find the inverse z-transform of $X(z) = \frac{z^{-4}}{z-1} + z^{-6} + \frac{z^{-3}}{z+0.5}$

Solution $x(n) = Z^{-1}\left(z^{-5}\frac{z}{z-1}\right) + Z^{-1}\left(z^{-6}\cdot 1\right) + Z^{-1}\left(z^{-4}\frac{z}{z+0.5}\right)$

Using Table ⇨ $x(n) = u(n-5) + \delta(n-6) + (-0.5)^{n-4}u(n-4)$

Inverse z-Transform: Using Partial Fraction

Example 11

Problem: Find the inverse z-transform of $X(z) = \dfrac{1}{(1 - z^{-1})(1 - 0.5z^{-1})}$

Solution: First eliminate the negative power of z.

$$X(z) = \frac{z^2}{z^2(1 - z^{-1})(1 - 0.5z^{-1})} = \frac{z^2}{(z-1)(z-0.5)}$$

Dividing both sides by z, $\dfrac{X(z)}{z} = \dfrac{z}{(z-1)(z-0.5)} = \dfrac{A}{(z-1)} + \dfrac{B}{(z-0.5)}$

Finding the constants:

$$A = (z-1)\frac{X(z)}{z}\Big|_{z=1} = \frac{z}{(z-0.5)}\Big|_{z=1} = 2$$

$$B = (z-0.5)\frac{X(z)}{z}\Big|_{z=0.5} = \frac{z}{(z-1)}\Big|_{z=0.5} = -1$$

$$\frac{X(z)}{z} = \frac{2}{(z-1)} + \frac{-1}{(z-0.5)}$$

multiplying by z

$$X(z) = \frac{2z}{(z-1)} + \frac{-z}{(z-0.5)}$$

Therefore, inverse z-transform is: $x(n) = 2u(n) - (0.5)^n u(n)$

Tech. Acc./ Manar Fathy

Inverse z-Transform: Using Partial Fraction

Problem: Find $y(n)$ if $Y(z) = \dfrac{z^2(z+1)}{(z-1)(z^2-z+0.5)}$ **Example 12**

Solution:

Dividing Y(z) by z, $$\frac{Y(z)}{z} = \frac{z(z+1)}{(z-1)(z^2-z+0.5)}$$

Applying the partial fraction expansion, ➡ $$\frac{Y(z)}{z} = \frac{B}{z-1} + \frac{A}{(z-0.5-j0.5)} + \frac{A^*}{(z-0.5+j0.5)}$$

We first find B: $$B = (z-1)\frac{Y(z)}{z}\bigg|_{z=1} = \frac{z(z+1)}{(z^2-z+0.5)}\bigg|_{z=1} = \frac{1\times(1+1)}{(1^2-1+0.5)} = 4$$

Next find A: $$A = (z-0.5-j0.5)\frac{Y(z)}{z}\bigg|_{z=0.5+j0.5} = \frac{z(z+1)}{(z-1)(z-0.5+j0.5)}\bigg|_{z=0.5+j0.5}$$

$$= \frac{(0.5+j0.5)(0.5+j0.5+1)}{(0.5+j0.5-1)(0.5+j0.5-0.5+j0.5)} = \frac{(0.5+j0.5)(1.5+j0.5)}{(-0.5+j0.5)j}$$

Tech. Acc./ Manar Fathy

Example 12 –contd.

Using the polar form, $A = \dfrac{(0.707\angle 45^\circ)(1.58114\angle 18.43^\circ)}{(0.707\angle 135^\circ)(1\angle 90^\circ)} = 1.58114\angle -161.57^\circ$

$$A^* = \bar{A} = 1.58114\angle 161.57^\circ.$$

$$P = 0.5 + 0.5j = |P|\angle\theta = 0.707\angle 45^0 \text{ and } P^* = |P|\angle -\theta = 0.707\angle -45^0$$

Now we have: $Y(z) = \dfrac{4z}{z-1} + \dfrac{Az}{(z-P)} + \dfrac{A^* z}{(z-P^*)}$

Therefore, the inverse z-transform is:

from Line 15 in Table

$$\begin{aligned} y(n) &= 4u(n) + 2|A|(|P|)^n \cos(n\theta + \varphi)\,u(n) \\ &= 4u(n) + 3.1623(0.7071)^n \cos(45^0 n - 161.57^0)\,u(n) \end{aligned}$$

Tech. Acc./ Manar Fathy

Inverse z-Transform: Using Partial Fraction

Example 13

Problem: Find $x(n)$ if $X(z)=\dfrac{z^2}{(z-1)(z-0.5)^2}$

Solution:

Dividing both sides by z: $\dfrac{X(z)}{z}=\dfrac{z}{(z-1)(z-0.5)^2}=\dfrac{A}{z-1}+\dfrac{B}{z-0.5}+\dfrac{C}{(z-0.5)^2}$

Where, $A=(z-1)\dfrac{X(z)}{z}\Big|_{z=1}=\dfrac{z}{(z-0.5)^2}\Big|_{z=1}=4$

Using the formulas for mth-order,

$$\frac{R_m}{(z-p)}+\frac{R_{m-1}}{(z-p)^2}+\cdots+\frac{R_1}{(z-p)^m}$$

$$R_k=\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left((z-p)^m\frac{X(z)}{z}\right)\Big|_{z=p}$$

$$B=R_2=\frac{1}{(2-1)!}\frac{d}{dz}\left\{(z-0.5)^2\frac{X(z)}{z}\right\}_{z=0.5}$$

m=2, p=0.5

$$=\frac{d}{dz}\left(\frac{z}{z-1}\right)\Big|_{z=0.5}=\frac{-1}{(z-1)^2}\Big|_{z=0.5}=-4$$

Tech. Acc./ Manar Fathy

Example 13 –contd.

$$C = R_1 = \frac{1}{(1-1)!}\frac{d^0}{dz^0}\left\{(z-0.5)^2\frac{X(z)}{z}\right\}_{z=0.5}$$

$$= \left.\frac{z}{z-1}\right|_{z=0.5} = -1$$

Then, $$X(z) = \frac{4z}{z-1} + \frac{-4z}{z-0.5} + \frac{-1z}{(z-0.5)^2}$$

From Table,

$$Z^{-1}\left\{\frac{z}{z-1}\right\} = u(n)$$

$$Z^{-1}\left\{\frac{z}{z-0.5}\right\} = (0.5)^n u(n)$$

$$Z^{-1}\left\{\frac{z}{(z-0.5)^2}\right\} = 2n(0.5)^n u(n)$$

Finally we get, $$x(n) = 4u(n) - 4(0.5)^n u(n) - 2n(0.5)^n u(n)$$

Tech. Acc./ Manar Fathy

Difference Equation Using Z-Transform

- The procedure to solve difference equation using Z-Transform:

  1. Apply the z-transform to the difference equation.
  2. Substitute the initial conditions.
  3. Solve for the difference equation in the z-transform domain.
  4. Find the solution in the time domain by applying the inverse z-transform.

Tech. Acc./ Manar Fathy

Example 17

Problem: Solve the difference equation when the initial condition is $y(-1) = 1$.

$$y(n) - 0.5y(n-1) = 5(0.2)^n u(n)$$

Solution:

We have

$$Z(y(n-1)) = \sum_{n=0}^{\infty} y(n-1)z^{-n}$$
$$= y(-1) + y(0)z^{-1} + y(1)z^{-2} + \cdots$$
$$= y(-1) + z^{-1}\left(y(0) + y(1)z^{-1} + y(2)z^{-2} + \cdots\right) \Rightarrow Z(y(n-1)) = y(-1) + z^{-1}Y(z)$$

Taking z-transform on both sides: $Y(z) - 0.5(y(-1) + z^{-1}Y(z)) = 5Z(0.2^n u(n))$

Substituting the initial condition and z-transform on right hand side using Table:

$$Y(z) - 0.5(1 + z^{-1}Y(z)) = 5z/(z-0.2)$$

Arranging Y(z) on left hand side: $Y(z) - 0.5z^{-1}Y(z) = 0.5 + 5z/(z-0.2)$

$$\Rightarrow Y(z) = \frac{(5.5z-0.1)}{(1-0.5z^{-1})(z-0.2)} = \frac{z(5.5z-0.1)}{(z-0.5)(z-0.2)} \Rightarrow \frac{Y(z)}{z} = \frac{5.5z-0.1}{(z-0.5)(z-0.2)} = \frac{A}{z-0.5} + \frac{B}{z-0.2}$$

Tech. Acc./ Manar Fathy

Example 17 –contd.

Solving for A and B:

$$A = (z-0.5)\frac{Y(z)}{z}\bigg|_{z=0.5} = \frac{5.5z-0.1}{z-0.2}\bigg|_{z=0.5} = \frac{5.5\times 0.5-0.1}{0.5-0.2} = 8.8333$$

$$B = (z-0.2)\frac{Y(z)}{z}\bigg|_{z=0.2} = \frac{5.5z-0.1}{z-0.5}\bigg|_{z=0.2} = \frac{5.5\times 0.2-0.1}{0.2-0.5} = -3.3333$$

Therefore, $Y(z) = \dfrac{8.8333z}{(z-0.5)} + \dfrac{-3.3333z}{(z-0.2)}$

Taking inverse z-transform, we get the solution:

$$\boxed{y(n) = 8.3333(0.5)^n u(n) - 3.3333(0.2)^n u(n)}$$

Tech. Acc./ Manar Fathy

Example 18

Problem:

A DSP system is described by the following differential equation with zero initial condition:

$$y(n) + 0.1y(n-1) - 0.2y(n-2) = x(n) + x(n-1)$$

a. Determine the impulse response $y(n)$ due to the impulse sequence $x(n) = \delta(n)$.

b. Determine the system response $y(n)$ due to the unit step function excitation, where $u(n) = 1 \; for \; n \geq 0$

Solution:

a. Applying the z-transform on both sides: $Y(z) + 0.1Y(z)z^{-1} - 0.2Y(z)z^{-2} = X(z) + X(z)z^{-1}$

Applying $X(z) = Z(\delta(n)) = 1$ On right side

$$\Rightarrow \quad Y(z)(1 + 0.1z^{-1} - 0.2z^{-2}) = 1(1 + z^{-1})$$

$$\Rightarrow \quad Y(z) = \frac{1 + z^{-1}}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

Tech. Acc./ Manar Fathy

Example 18 –contd.

We multiply the numerator and denominator by z^2

$$Y(z) = \frac{z^2+z}{z^2+0.1z-0.2} = \frac{z(z+1)}{(z-0.4)(z+0.5)}$$

Using the partial fraction expansion

$$\frac{Y(z)}{z} = \frac{z+1}{(z-0.4)(z+0.5)} = \frac{A}{z-0.4} + \frac{B}{z+0.5}$$

Solving for A and B:

$$A = (z-0.4)\frac{Y(z)}{z}\bigg|_{z=0.4} = \frac{z+1}{z+0.5}\bigg|_{z=0.4} = \frac{0.4+1}{0.4+0.5} = 1.5556$$

$$B = (z+0.5)\frac{Y(z)}{z}\bigg|_{z=-0.5} = \frac{z+1}{z-0.4}\bigg|_{z=-0.5} = \frac{-0.5+1}{-0.5-0.4} = -0.5556$$

Therefore,

$$Y(z) = \frac{1.5556z}{(z-0.4)} + \frac{-0.5556z}{(z+0.5)}$$

Hence the impulse response:

$$y(n) = 1.5556(0.4)^n u(n) - 0.5556(-0.5)^n u(n)$$

Tech. Acc./ Manar Fathy

Example 18 –contd.

b. The input is step unit function: $x(n) = u(n)$

Corresponding z-transform: $X(z) = \frac{z}{z-1}$

Notice that $Y(z) + 0.1\,Y(z)z^{-1} - 0.2\,Y(z)z^{-2} = X(z) + X(z)z^{-1}$ [Slide 21]

Then the z-transform of the output sequence $y(n)$,

$$Y(z) = \left(\frac{z}{z-1}\right)\left(\frac{1+z^{-1}}{1+0.1z^{-1}-0.2z^{-2}}\right) = \frac{z^2(z+1)}{(z-1)(z-0.4)(z+0.5)}$$

Using the partial fraction expansion $Y(z) = \frac{2.2222z}{z-1} + \frac{-1.0370z}{z-0.4} + \frac{-0.1852z}{z+0.5}$

and the system response is found by using Table:

$$y(n) = 2.2222u(n) - 1.0370(0.4)^n u(n) - 0.1852(-0.5)^n u(n)$$